Relations and Their Properties.

Def. Let A and B be sets. A binary relation from A to B is a subset of \( A \times B \).

**If** \((a, b) \in R\) we say \(aRb\).
**If** \((a, b) \notin R\) we say \(a \notR b\).

Ex. Let \(A = \) cities in U.S. Let \(B = \) states in U.S.
Define a relation \(R\) from \(A\) to \(B\) by
\(aRb\) if \(a\) "is a city in the state" \(b\).
(\(Miami, FL\) \(\in R\) \(\) (\(Miami, GA\) \(\not\in R\)

Def. A relation from \(A\) to \(A\) is called a relation on \(A\) (subset of \(A \times A\)).

Ex. Consider the relations on \(\mathbb{Z}\):
\(R_1 = \{(a, b) \mid a \leq b\} = \{(-2, 5), (3, 3), (-2, 0), (1, 8), \ldots\}
\(R_2 = \{(a, b) \mid a = b\) or \(a = -b\} = \{(0, 0), (1, 1), (1, -1), (3, 3), (5, -5), \ldots\}
\(R_3 = \{(a, b) \mid a + b \leq 3\} = \{(1, 2), (-8, 6), (-6, -4), \ldots\}

Ex. How many relations are there on a set with \(n\) elements?
\(|A| = n\) \(\Rightarrow |A \times A| = n^2\) \(\Rightarrow |P(A \times A)| = 2^{n^2}\)

Def. Properties of a relation \(R\) on \(A\):

1. Reflexive \((a, a) \in R\) for all \(a \in A\) \((aRa)\)
2. Irreflexive \((a, a) \notin R\) for all \(a \in A\) \((a \notR a)\)
3. Symmetric If \((a, b) \in R\) then \((b, a) \in R\) for all \(a, b \in A\) \((If \ aRb \ then \ bRa \ for \ all \ a, b \in A)\)
4. **Antisymmetric** If \((a,b)\in R\) and \((b,a)\in R\) then \(a=b\) for all \(a,b\in A\) (If \(a\neq b\) and \(b\neq a\) then \(a=b\) for all \(a,b\in A\))

5. **Transitive** If \((a,b)\in R\) and \((b,c)\in R\) then \((a,c)\in R\) for all \(a,b,c\in A\) (If \(a\neq b\) and \(b\neq c\) then \(a\neq c\) for all \(a,b,c\in A\))

**Examples**

1. \(A = \mathbb{N} = \{0,1,2,\ldots\} \) \(R = \{(a,b) \mid a+b \text{ even}\} = \{(0,2), (1,3), (2,4), \ldots\}\)

   - Reflexive: \((a,a)\in R\) for all \(a\in \mathbb{N}\) since \(a+a = 2a\) (even)
   - Not irreflexive: \((1,1)\in R\)
   - Symmetric: If \((a,b)\in R\) then \(a+b = 2z\), for some \(z\in \mathbb{Z}\), then \(b+a = 2z\) or \((b,a)\in R\) for all \(a,b\in \mathbb{Z}\)
   - Not antisymmetric: \((2,4)\in R\) and \((4,2)\in R\)

   - Transitive: If \((a,b)\in R\) and \((b,c)\in R\) then \(a+b\) is even and \(b+c\) is even, so either all \(a, b, c\) are even or all \(a, b, c\) is odd. Either way \(a+c\) is even or \((a,c)\in R\) for all \(a,b,c\in \mathbb{N}\)

2. \(A = \{1,2,3,4,5\}\)

   \(R = \{(1,1), (1,3), (4,2), (2,4), (2,3), (3,1)\}\)

   - Not reflexive since \((2,2)\notin R\)
   - Not irreflexive since \((1,1)\in R\)
   - Not symmetric since \((2,3)\in R\) but \((3,2)\notin R\)
   - Not transitive since \((4,2)\in R\) and \((2,4)\in R\) but \((4,4)\notin R\)
     or \((4,2)\in R\) and \((2,3)\in R\) but \((4,3)\notin R\)
     etc.
$3 \ A = \mathbb{Z} \ \\
R = \{(x,y) \mid x < y \equiv 0 \mod{3}\} \ \\
\text{Not reflexive: since } (1,1) \notin R \\
\text{Not irreflexive: since } (3,3) \in R \\
\text{Symmetric: If } (a,b) \in R \text{ then } a+b = 3z, \text{ for some } z \in \mathbb{Z} \text{, then } b+a = 3z \text{ or } (b,a) \in R \text{ for all } a, b \in \mathbb{Z} \\
\text{Not antisymmetric: since } (1,2) \in R \text{ and } (2,1) \in R \\
\text{Not transitive: Since } (1,2) \in R \text{ and } (2,1) \in R \text{ but } (1,1) \notin R$ \\

$A = \mathbb{Z} \ \\
R = \{(x,y) \mid x < y \equiv 0 \mod{3}\} \ \\
\text{Not reflexive: since } (1,1) \notin R \\
\text{Irreflexive: } (a,a) \notin R \text{ for all } a \in \mathbb{Z} \text{ since } a \neq a \\
\text{Not symmetric } (1,3) \in R \text{ but } (3,1) \notin R \\
\text{Antisymmetric: If } (a,b) \in R \text{ and } (b,a) \in R \text{ then } a < b \text{ and } b < a \text{ (impossible)} \text{ so property holds. } (\rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow) \\
\text{Transitive: If } (a,b) \in R \text{ and } (b,c) \in R \text{ then } a < b \text{ and } b < c \text{ so } a < c \text{ so } (a,c) \in R \text{ for all } a, b, c \in \mathbb{Z} \text{ (called the transitive property of inequality)}$
5. \( A = \mathbb{Z} \) \( R = \{ (x,y) \mid x \leq y \} \)

\( R = \{ (-3,5), (1,1), (4,12) \} \)

Reflexive: \((a,a) \in R\) for all \(a \in \mathbb{Z}\) since \(a \leq a\)

Not irreflexive: \((1,1) \notin R\)

Not symmetric: \((3,6) \in R\) but \((6,3) \notin R\)

Antisymmetric: If \((a,b) \in R\) and \((b,a) \in R\) then \(a \leq b\) and \(b \leq a\) so \(a = b\) for all \(a, b \in \mathbb{Z}\)

\[ \text{(Ex) } \]

6a) \( A = P = \{1,2,3, \ldots \} \)

\[ R = \{ (x,y) \mid x \text{ divides } y, \ x \mid y \text{ or } \frac{y}{x} = \mathbb{Z} \text{ for } z \in \mathbb{Z} \} \]

Reflexive: \((a,a) \in R\) for all \(a \in P\) since \(\frac{a}{a} = 1\)

Not irreflexive: \((1,1) \notin R\)

Not symmetric: \((2,4) \in R\) but \((4,2) \notin R\)

Antisymmetric: If \((a,b) \in R\) and \((b,a) \in R\) then \(\frac{b}{a} = z_1\) and \(\frac{a}{b} = z_2\), \(z_1, z_2 \in \mathbb{Z}\)

So \(b = a z_1\) and \(a = b z_2\) or

\(b = b z_1 z_2\) and \(z_1 z_2 = 1\) so

\(z_1 = 1\) or \(z_1 = -1\) and

But \(a, b \in P\) so \(z_1 = z_2 = 1\) and

\(a = b\).

Transitive: If \((a,b) \in R\) and \((b,c) \in R\) then

\(\frac{b}{a} = z_1\) and \(\frac{c}{b} = z_2\) or

\(b = a z_1\) and \(b = c z_2\) so \(a \in \mathbb{Z}\) for all \(a, b, c \in \mathbb{Z}\) and

\(a, b, c \in \mathbb{Z}\)
(7) \[ A = \{1,2,3\} \]
\[ R = \{ (1,1), (2,2), (3,3) \} \]
Reflexive, Not irreflexive \((1,1) \in R\),
Symmetric, Antisymmetric, Transitive

(8) \[ A = \{1,2,3\} \]
\[ R = \emptyset \]
Not reflexive \((1,1) \notin R\), Irreflexive, Symmetric
Antisymmetric, Transitive

(9) \[ A = \emptyset \]
\[ R = \emptyset \]
Reflexive: Assume \( R \) is not reflexive on \( A \)
there is an \( a \in A \) with \( (a,a) \notin R \)
Contradiction
Irreflexive Assume \( R \) is not irreflexive on \( A \)
there is an \( a \in A \) with \( (a,a) \in R \)
Contradiction
Symmetric
Antisymmetric
Transitive

(10) \[ A = \text{people in the world} \]
\[ R = \{ (a,b) | a \text{ is a parent of } b \} \]
Not reflexive: No one is their own parent
Irreflexive: Everyone is not their own parent
Not symmetric: If Bob is parent of John, then
John is not parent of Bob.
Antisymmetric: \( a \) cannot be parent of \( b \) and
\( b \) parent of \( a \) at same time
\[ f \rightarrow ? = 1 \]
Not transitive If \( a \) parent of \( b \) and \( b \) parent of \( c \)
then \( a \) is grandparent of \( c \)
Combining Relations

Ex. Let \( A = \{1,2,3,4,5\}\),
\( R_1 = \{(1,1), (2,2), (3,3)\} \quad R_2 = \{(1,2), (1,3), (1,4)\} \)
\( R_1 \cup R_2 = \{(1,1), (1,2), (1,3), (1,4), (2,2), (3,3)\} \)
\( R_1 \cap R_2 = \{(1,1)\} \)
\( R_1 - R_2 = \{(2,2), (3,3)\} \)
\( R_2 - R_1 = \{(1,2), (1,3), (1,4)\} \)

Def. Let \( R \) be a relation from a set \( A \) to a set \( B \).
- The inverse relation \( R^{-1} = \{(b,a) \mid (a,b) \in R\} \)
- The complementary relation \( \overline{R} = \{(a,b) \mid (a,b) \notin R\} \)

Ex. Let \( A = \mathbb{Z} \),
\( R = \{(a,b) \mid a < b\} \)
\( R^{-1} = \{(a,b) \mid a > b\} \)
\( \overline{R} = \{(a,b) \mid a \geq b\} \)

Thm. Let \( R \) be a relation on \( A \).
Prove \( R \) is symmetric iff \( R = R^{-1} \).

a) If \( R \) is symmetric then \( R = R^{-1} \)
\( (x,y) \in R \overset{\text{given}}{\iff} (y,x) \in R \overset{\text{sym}}{\iff} (x,y) \in R^{-1} \)

b) If \( R = R^{-1} \) then \( R \) is symmetric on \( A \)
\( (x,y) \in R \overset{\text{given}}{\iff} (x,y) \in R^{-1} \overset{\text{Def.}}{\iff} (y,x) \in R \)
Equivalence Relations

Def. An equivalence relation on a set $A$ is a relation which is reflexive, symmetric and transitive.

Ex1. $R$ is a relation on the set of strings of letters \{a,b,...,z\} such that $w_1 R w_2$ iff $l(w_1) = l(w_2)$ where $l(w)$ is the length of string $w$.

Reflexive: $w R w$ for all strings $w$ as $l(w) = l(w)$
Symmetric: If $w_1 R w_2$ then $l(w_1) = l(w_2)$, so $l(w_2) = l(w_1)$ and $w_2 R w_1$ for all strings $w_1, w_2$.
Transitive: If $w_1 R w_2$ and $w_2 R w_3$ then $l(w_1) = l(w_2) = l(w_3)$ so $l(w_1) = l(w_3)$ or $w_1 R w_3$ for all strings $w_1, w_2, w_3$.

Ex2. $R$ is a relation on the set of real numbers such that $x R y$ iff $x - y \in \mathbb{Z}$

E.g. $3.25 R 2.25$, $-8 R 6$, $\pi R \pi + 1$
Reflexive: $x R x$ since $x - x = 0 \in \mathbb{Z}$ for all real numbers $x$
Symmetric: If $x R y$ then $x - y = z$, $z \in \mathbb{Z}$, so $y - x = -z$, $-z \in \mathbb{Z}$, $y R x$ for all real numbers $x, y$
Transitive: If $x R y$ and $y R z$ then $x - y = z$, $y - z = w$ for $z, w \in \mathbb{Z}$ then $x - z = (z + w)$, $z + w \in \mathbb{Z}$ so $x R z$ for all real numbers $x, y, z$.

Ex3. $R$ is a relation on $\mathbb{Z}$

$R = \{(a, b) | a \equiv b \mod 2\}$ (a-b=2z, z\in\mathbb{Z})

Reflexive (a,a)\in R since a-a=0=2(0) for all a\in\mathbb{Z}
Symmetric If (a,b)\in R then a-b=2z, z\in\mathbb{Z} then b-a=2(-z) so (b,a)\in R for all a,b\in\mathbb{Z}
Transitive If (a,b)\in R and (b,c)\in R then a-b=2z, z\in\mathbb{Z}, b-c=2z, z\in\mathbb{Z} then a-c=2(z+z) so (a,c)\in R for all a,b,c\in\mathbb{Z}
Note: In EX3, can interchange the mod 2 for mod 3, mod 4; and will still have an equivalence relation.

Def. let $R$ be an equivalence relation on a set $A$. The equivalence class of $a \in A$ is the set of all elements that are related to "$a$" denoted $[a]$. 

$$[a] = \{b \mid bRa\} = \{b \mid (b,a) \in R\}.$$ 

Equivalence classes for EX1, EX2, EX3

**EX1.** $[\lambda] = \{\lambda\}$, $\lambda$ = empty string

$[a] = \{a, b, c, \ldots \} \subset \mathbb{Z}$, $|a| = 26$

$[aa] = \{aa, ab, \ldots \} \subset \mathbb{Z}$, $|aa| = 26^2$

$[aaa] = \{aaa, aab, \ldots \} \subset \mathbb{Z}$, $|aaa| = 26^3$

infinite number of equivalence classes

**EX2.** $[0] = \{\ldots, -3, -2, -1, 0, 1, 2, \ldots \} \subset \mathbb{Z}$, $|0| = \{0 + 2|z \in \mathbb{Z}\}$

$[2.25] = \{\ldots, -1.75, -1.25, 1.25, 2.25, 3.25, \ldots \} \subset \mathbb{Z}$, $|2.25| = \{2.25 + 2|z \in \mathbb{Z}\}$

$[\pi] = \{\ldots, \pi - 1, \pi, \pi + 1, \pi + 2, \ldots \} \subset \mathbb{Z}$, $|\pi| = \{\pi + 2|z \in \mathbb{Z}\}$

infinite number of equivalence classes

**EX3.** $[0] = \{\ldots, -4, -2, 0, 2, 4, \ldots \} \subset \mathbb{Z}$

$[1] = \{\ldots, -3, -1, 1, 3, 5, 7, \ldots \} \subset \mathbb{Z}$

$2$ equivalence classes

Let $A = \mathbb{Z}$, $R = \{(a,b) \mid a \equiv b \mod 4\}$. is an equivalence relation.

$[0] = \{\ldots, -8, -4, 0, 4, 8, \ldots \} \subset \mathbb{Z}$

$[1] = \{\ldots, -7, -3, 1, 5, 9, \ldots \} \subset \mathbb{Z}$

$[2] = \{\ldots, -6, -2, 2, 6, 10, \ldots \} \subset \mathbb{Z}$

$[3] = \{\ldots, -5, -1, 3, 7, 11, \ldots \} \subset \mathbb{Z}$

4 equivalence classes.

In general, $R = \{(a,b) \mid a \equiv b \mod 3\}$ will have $p$ equivalence classes.
Partition

An equivalence relation $R$ on $A$ partitions the set $A$ into equivalence classes. Every element of $A$ is in one and only one equivalence class.

$A \quad [a] = \{b | b Ra\}$

Above is proved with the following theorem:

**Theorem:** Let $R$ be an equivalence relation on a set $A$. The following statements are equivalent:

i) $a R b$

ii) $[a] = [b]$

iii) $[a] \cap [b] \neq \emptyset$

**Proof.**

i$\Rightarrow$ii: If $a R b$ then $[a] = [b]$

- Take $x \in [a]$ \stackrel{\text{Def}[a]}{\Rightarrow} xRa
- Take $x \in [b]$ \stackrel{\text{Def}[b]}{\Rightarrow} xRb

\[ (\text{Given } a R b) \quad \Rightarrow \quad xRb \Rightarrow x \in [b] \]

\[ \text{Ref} \]

\[ xRb, bRa \Rightarrow xRb \Rightarrow x \in [a] \]

\[ \text{Sym} \]

So $[a] = [b]$

ii$\Rightarrow$iii: If $[a] = [b]$ then $[a] \cap [b] \neq \emptyset$

Since $R$ is reflexive $a \in [a] = [b]$ so $a \in [a] \cap [b] \neq \emptyset$

iii$\Rightarrow$i: If $[a] \cap [b] \neq \emptyset$ then $a R b$

- Let $x \in [a] \cap [b]$ \stackrel{\text{Def}}{\Rightarrow} xRa and $xRb$
- \[ \text{Sym} \]
- \[ aRx \text{ and } xRb \]
- \[ \text{Trans} \]
- \[ aRb \]