Relations and their Properties.

**Def.** Let $A$ and $B$ be sets. A binary relation from $A$ to $B$ is a subset of $A \times B$.

If $(a, b) \in R$ we say $a R b$.
If $(a, b) \notin R$ we say $a \not R b$.

**Ex.** Let $A =$ cities in U.S. Let $B =$ states in U.S.
Define a relation $R$ from $A$ to $B$ by:

$a R b$ if $a$ "is a city in the state" $b$.

$(\text{Miami, FL}) \in R \quad (\text{Miami, CA}) \notin R$

**Def.** A relation from $A$ to $A$ is called a relation on $A$ (subset of $A \times A$).

**Ex.** Consider the relations on $\mathbb{Z}$:

- $R_1 = \{(a, b) \mid a \leq b\} = \{(-2, 5), (3, 3), (-2, 4), (1, 8), \ldots\}$
- $R_2 = \{(a, b) \mid a = b$ or $a = -b\} = \{(0, 0), (1, 1), (1, -1), (2, 2), (2, -2), \ldots\}$
- $R_3 = \{(a, b) \mid a + b \leq 3\} = \{(1, 2), (-2, 6), (-2, -4), \ldots\}$

**Ex.** How many relations are there on a set with $n$ elements?

$|A| = n \quad |A \times A| = n^2 \quad |P(A \times A)| = 2^{n^2}$

**Def.** Properties of a relation $R$ on $A$.

1. Reflexive $(a, a) \in R$ for all $a \in A$ ($aRa$)
2. Irreflexive $(a, a) \notin R$ for all $a \in A$ ($a \not R a$)
3. Symmetric If $(a, b) \in R$ then $(b, a) \in R$ for all $a, b \in A$ (If $a R b$ then $b R a$ for all $a, b \in A$)
(3) $A = \mathbb{Z}$
$R = \{(x,y) | x+y \equiv 0 \pmod{3}\}$

**Notation:**
$x \equiv y \pmod{p} \iff x-y$ is a multiple of $p$ for some $z \in \mathbb{Z}$

$R = \{(0,3), (-1,4), (-2,-2), (1,2), (3,3), \ldots \}$

- **Not reflexive:** since $(1,1) \notin R$
- **Not irreflexive:** since $(3,3) \in R$
- **Symmetric:** If $(a,b) \in R$ then $a+b = 3z$, for some $z \in \mathbb{Z}$, then $b+a = 3z$ or $(b,a) \in R$ for all $a, b \in \mathbb{Z}$
- **Not antisymmetric:** since $(1,2) \in R$ and $(2,1) \in R$
- **Not transitive:** since $(1,2) \in R$ and $(2,1) \in R$ but $(1,1) \notin R$

(4) $A = \mathbb{Z}$
$R = \{(x,y) | x < y \pmod{3}\} = \{(-2,3), (3,4), \ldots \}$

- **Not reflexive:** $(1,1) \notin R$
- **Irreflexive:** $(a,a) \notin R$ for all $a \in \mathbb{Z}$ since $a \neq a$
- **Not symmetric:** $(1,3) \notin R$ but $(3,1) \in R$
- **Antisymmetric:** If $(a,b) \in R$ and $(b,a) \in R$ then $a < b$ and $b < a$ (impossible) so property holds.

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**Logic Table**

Transitive: If $(a,b) \in R$ and $(b,c) \in R$
then $a < b$ and $b < c$ so $a < c$

$a, (ac) \in R$ for all $a, b, c \in \mathbb{Z}$
(called the transitive property of inequality)
4. Antisymmetric If \((a,b) \in R\) and \((b,a) \in R\) then \(a = b\)
for all \(a, b \in A\). (If \(a \neq b\) and \(b \neq a\) then \(a = b\) for all \(a, b \in A\))

5. Transitive If \((a,b) \in R\) and \((b,c) \in R\) then \((a,c) \in R\)
for all \(a, b, c \in A\). (If \(a \neq b\) and \(b \neq c\) then \(a \neq c\) for all \(a, b, c \in A\))

Examples

1. \(A = \mathbb{N} = \{0, 1, 2, \ldots\}\)
   \(R = \{(a,b) \mid a + b \text{ even}\}\)
   \(= \{(0,2), (1,3), (2,4), \ldots\}\)

   Reflexive \((a,a) \in R\) for all \(a \in \mathbb{N}\) since \(a + a = 2a\) (even)

   Not irreflexive \((0,0) \notin R\)

   Symmetric If \((a,b) \in R\) then \(a + b = 2z\), for some \(z \in \mathbb{Z}\), then
   \(b + a = 2z\) or \((b,a) \in R\) for all \(a, b \in \mathbb{Z}\)

   Not antisymmetric \((2,4) \in R\) and \((4,2) \in R\)

   Transitive If \((a,b) \in R\) and \((b,c) \in R\) then \(a + b\) is even
   and \(b + c\) is even, so either all \(a, b, c\) are even or all \(a, b, c\) are odd.
   Either way \(a + c\) is even or \((a,c) \in R\) for all \(a, b, c \in \mathbb{N}\)

2. \(A = \{1, 2, 3, 4, 5\}\)

   \(R = \{(1,1), (1,3), (4,2), (2,4), (2,3), (3,1)\}\)

   Not reflexive since \((2,2) \notin R\)

   Not irreflexive since \((1,1) \in R\)

   Not symmetric since \((2,3) \in R\) but \((3,2) \notin R\)

   Not transitive since \((4,2) \in R\) and \((2,4) \in R\) but \((4,4) \notin R\)
   or \((4,2) \in R\) and \((2,3) \in R\) but \((4,3) \notin R\)

   etc.
3. \( A = \mathbb{Z} \) \( R = \{ (x,y) \mid x \leq y \} \)

\( R = \{ (-5,5), (1,1), (4,12) \} \)

- Reflexive: \((a,a) \in R\) for all \(a \in \mathbb{Z}\) since \(a \leq a\)
- Not irreflexive: \((1,1) \in R\)
- Not symmetric: \((3,4) \in R\) but \((4,3) \notin R\)
- Antisymmetric: If \((a,b) \in R\) and \((b,a) \in R\) then \(a \leq b\) and \(b \leq a\) so \(a = b\) for all \(a, b \in \mathbb{Z}\)

\[ \begin{align*}
\text{Example}\; & 6a) \quad A = P = \{1,2,\ldots,3\} \\
& R = \{ (x,y) \mid x \text{ divides } y, x \mid y \text{ or } \frac{y}{x} = \mathbb{Z} \text{ for } x \in \mathbb{Z} \} \\
& a) R = \{ (2,4), (3,6), (6,1) \} \\
& \text{Reflexive: } (a,a) \in R \text{ for all } a \in A \\
& \text{Since } \frac{a}{a} = 1 \\
& \text{Not irreflexive: } (1,1) \notin R \\
& \text{Not symmetric: } (a,b) \in R, (b,a) \in R \\
& \text{Antisymmetric: If } (a,b) \in R \text{ and } (b,a) \in R \\
& \text{then } \frac{a}{b} = z_1 \text{ and } \frac{b}{a} = z_2, z_1, z_2 \in 2 \\
& \text{So } b = a z_1 \text{ and } a = b z_2 \text{ or } \\
& b = b z_2 \text{ or } z_2 = 1 \text{ so } \\
& z_1 = z_2 = 1 \text{ or } z_2 = z_2 = -1 \\
& \text{But } a, b \in P \text{ so } z_1, z_2 \in 1 \text{ and } \frac{a}{b} = 1 \text{ and } \frac{b}{a} = 1 \\
& \text{Not antisymmetric: } (5,5) \in R \text{ and } (5,5) \in R \\
\end{align*} \]

\[ \begin{align*}
\text{Example}\; & 6b) \quad A = \mathbb{Z} \\
& R = \{ (x,y) \mid (x,y) \in \mathbb{Z} \} \\
& b) R = \{ (2,4), (1,-1), (-5,25), (80) \} \\
& \text{Not reflexive: } \frac{a}{a} \text{ undefined} \\
& \text{Not irreflexive: } (-2,-2) \in R \\
& \text{Not symmetric: } (-28) \in R, (8,7) \in R \\
& \text{Not antisymmetric: } (5,5) \in R \text{ and } (5,5) \in R \\
\end{align*} \]
(i) \( A = \{1, 2, 3\} \)
\( R = \{(1,1), (2,2), (3,3)\} \)
Reflexive, Not Irreflexive \((1,1) \notin R\), Symmetric, Antisymmetric, Transitive

(ii) \( A = \{1, 2, 3\} \)
\( R = \emptyset \)
Not Reflexive \((1,1) \notin R\), Irreflexive, Symmetric, Antisymmetric, Transitive

(iii) \( A = \emptyset \)
\( R = \emptyset \)

Reflexive: Assume \( R \) is not reflexive on \( A \)
there is an \( a \in A \) with \( (a,a) \notin R \)

Contradiction

Irreflexive: Assume \( R \) is not irreflexive on \( A \)
there is an \( a \in A \) with \( (a,a) \in R \)

Contradiction

Symmetric
Antisymmetric
Transitive

(iv) \( A = \text{people in the world} \)
\( R = \{(a,b) \mid a \text{ is parent of } b \} \)

Not reflexive: No one is their own parent
Irreflexive: Everyone is not their own parent
Not symmetric: If Bob is parent of John, then John is not parent of Bob.
Antisymmetric: \( a \) cannot be parent of \( b \) and \( b \) parent of \( a \) at same time
\( \exists a_0. a_0 \to b \land b \to a_0 \land a_0 \neq b \)

Not transitive: If a parent of \( b \) and \( b \) parent of \( c \) then \( a \) is grandparent of \( c \)
Combining Relations

Ex. Let $A = \{1, 2, 3, 4\}$
$R_1 = \{(1,1), (2,2), (3,3)\}$ $R_2 = \{(1,1), (1,2), (1,3), (1,4)\}$
$R_1 \cup R_2 = \{(1,1), (1,2), (1,3), (1,4), (2,2), (3,3)\}$
$R_1 \cap R_2 = \{(1,1)\}$
$R_1 - R_2 = \{(2,2), (3,3)\}$
$R_2 - R_1 = \{(1,2), (1,3), (1,4)\}$

Def. Let $R$ be a relation from a set $A$ to a set $B$.

- The inverse relation $R^{-1} = \{(b,a) \mid (a,b) \in R\}$
  (a relation from set $B$ to set $A$)
- The complementary relation $\overline{R} = \{(a,b) \mid (a,b) \notin R\}$
  (a relation from set $A$ to set $B$, $\mathcal{U} = A \times B$)

Ex.
$A = \mathbb{Z}$
$R = \{(a,b) \mid a < b\}$
$R^{-1} = \{(a,b) \mid a > b\}$
$\overline{R} = \{(a,b) \mid a \geq b\}$

Thm. Let $R$ be a relation on $A$.

Prove $R$ is symmetric iff $R = R^{-1}$

a) If $R$ is symmetric then $R = R^{-1}$

\[ (x, y) \in R \overset{\text{sym}}{\iff} (y, x) \in R \overset{\text{Def. } R^{-1}}{\iff} (x, y) \in R^{-1} \]

b) If $R = R^{-1}$ then $R$ is symmetric on $A$

\[ (x, y) \in R \overset{\text{Def. } R^{-1}}{\iff} (x, y) \in R^{-1} \overset{\text{Def. } R} {\iff} (y, x) \in R \]

Equivalence Relations

Def. An equivalence relation on a set $A$ is a relation which is reflexive, symmetric and transitive.

Ex1. $R$ is a relation on the set of strings of letters $\{a, b, \ldots, z\}$ such that $w_1 R w_2$ iff $l(w_1) = l(w_2)$ where $l(w)$ is the length of string $w$.

- Reflexive: $w R w$ for all strings $w$ as $l(w) = l(w)$
- Symmetric: If $w_1 R w_2$ then $l(w_1) = l(w_2)$, so $l(w_2) = l(w_1)$ and $w_2 R w_1$ for all strings $w_1, w_2$.
- Transitive: If $w_1 R w_2$ and $w_2 R w_3$ then $l(w_1) = l(w_2) = l(w_3)$ so $l(w_1) = l(w_3)$ or $w_1 R w_3$ for all strings $w_1, w_2, w_3$.

Ex2. $R$ is a relation on the set of real numbers such that $x R y$ iff $x - y \in \mathbb{Z}$

- Reflexive: $x R x$ since $x - x = 0 \in \mathbb{Z}$ for all real numbers $x$
- Symmetric: If $x R y$ then $x - y = z \in \mathbb{Z}$, so $y - x = -z \in \mathbb{Z}$.
- Transitive: If $x R y$ and $y R z$ then $x - y = z_1 \in \mathbb{Z}$ and $y - z = z_2 \in \mathbb{Z}$, so $x - z_1 = z_2 \in \mathbb{Z}$ for all real numbers $x, y, z$.

Ex3. $R$ is a relation on $\mathbb{Z}$

- Reflexive: $(a, b) R (a, b)$ since $a - a = 0 \in \mathbb{Z}$ for all $a, b \in \mathbb{Z}$
- Symmetric: If $(a, b) R (c, d)$ then $a - b = 2z \in \mathbb{Z}$ and $c - d = 2z \in \mathbb{Z}$ so $b - a = 2(-z) \in \mathbb{Z}$ for all $a, b, c, d \in \mathbb{Z}$
- Transitive: If $(a, b) R (c, d)$ and $(b, c) R (e, f)$ then $a - b = 2z_1 \in \mathbb{Z}$, $b - c = 2z_2 \in \mathbb{Z}$, so $a - c = 2(z_1 + z_2) \in \mathbb{Z}$ for all $a, b, c, d, e, f \in \mathbb{Z}$.
Note: In EX3, can interchange the mod2 for mod3, mod4, and will still have an equivalence relation.

Def. Let R be an equivalence relation on a set A. The equivalence class of a \( a \in A \) is the set of all elements that are related to "a" denoted \([a]\).

\([a] = \{ b \mid (b,a) \in R \}\)

Equivalence classes for EX1, EX2, EX3

**EX1.**

\([\lambda] = \{ \lambda \} \quad \lambda = \text{empty string} \)

\([a] = \{ a, 10, c, \ldots \} \quad \# 3 \quad |[a]| = 26 \)

\([aa] = \{ aa, ab, \ldots \} \quad \# 3 \quad |[aa]| = 26 \)

\([aaa] = \{ aaa, \ldots \} \quad \# 3 \quad |[aaa]| = 26 \)

infinite number of equivalence classes

**EX2.**

\([0] = \{ \ldots, 3, 2, 1, 0, 1, 2, \ldots \} \quad \# 3 \quad \{0+2 \mid 2 \in \mathbb{Z}\} \)

\([3.25] = \{ \ldots, -1.75, -1.25, 0.25, 1.25, 2.25, 3.25, \ldots \} \quad \# 3 \quad \{3.25+2 \mid 2 \in \mathbb{Z}\} \)

\([\pi] = \{ \ldots, \pi-1, \pi, \pi+1, \pi+2, \ldots \} \quad \# 3 \quad \{\pi+2 \mid 2 \in \mathbb{Z}\} \)

infinite number of equivalence classes

**EX3.**

\([0] = \{ \ldots, -4, -2, 0, 2, 4, 6, \ldots \} \quad \# 3 \)

\([1] = \{ \ldots, -2, -1, 1, 3, 5, 7, \ldots \} \quad \# 3 \)

infinite equivalence classes

Let \( A = \mathbb{Z} \), \( R = \{ (a, b) \mid a \equiv b \mod 43 \} \) is an equivalence relation.

\([0] = \{ \ldots, 44, 0, 4, 8, \ldots \} \quad \# 3 \)

\([1] = \{ \ldots, -3, -1, 1, 3, 5, 7, \ldots \} \quad \# 3 \)

\([2] = \{ \ldots, -1, 3, 7, 11, \ldots \} \quad \# 3 \)

\([3] = \{ \ldots, 5, -1, 3, 7, 11, \ldots \} \quad \# 3 \)

4 equivalence classes.

\( \text{and } \sum_{n=0}^{43} b \mod 43 \) will have
Partition

An equivalence relation $R$ on $A$ partitions the set $A$ into equivalence classes. Every element of $A$ is in one and only one equivalence class.

$A$ \[ [a] = \{ b | bRa \} \]

Above is proved with the following theorem:

Thm: Let $R$ be an equivalence relation on a set $A$. The following statements are equivalent:

i) $aRb$

ii) $[a] = [b]$

iii) $[a] \cap [b] \neq \emptyset$

Proof:

i $\Rightarrow$ ii: If $aRb$ then $[a] = [b]$

Take $x \in [a]$ by definition $xRa$ (given $aRb$) $\Rightarrow xRb \Rightarrow x \in [b]$

Take $x \in [b]$ by definition $xRb$ (given $aRb$) $\Rightarrow xRa \\ \Rightarrow x \in [a]$

so $[a] = [b]$

ii $\Rightarrow$ iii: If $[a] = [b]$ then $[a] \cap [b] \neq \emptyset$

Since $R$ is reflexive $a \in [a] = [b]$ so $a \in [a] \cap [b] \neq \emptyset$

iii $\Rightarrow$ i: If $[a] \cap [b] \neq \emptyset$ then $aRb$

let $x \in [a] \cap [b]$ by definition $xRb$ and $xRa$

$\Rightarrow aRx$ and $xRb$

$\Rightarrow aRb$