Set Theory

Def. Set: a collection of well defined objects
Objects in the set are called elements or members
Sets - denoted by uppercase letters
Elements - denoted by lowercase letters
\( x \in A \), \( x \not\in A \)

Def. \( P = \{1, 2, 3\ldots\} \) 3 positive integers
\( N = \{0, 1, 2, \ldots\} \) 3 natural numbers
\( Z = \{\ldots -2, -1, 0, 1, 2, \ldots\} \) 3 integers
\( Q = \) rational numbers = \( \left\{ \frac{m}{n}, m, n \in \mathbb{Z}, n \neq 0 \right\} \)
\( \mathbb{R} = \) real numbers

Description of Sets
1. Describe the properties of elements of the set
   \( A = \{x \mid x \in N \text{ and } x \leq 5\} \)
2. List elements
   \( A = \{0, 1, 2, 3, 4, 5\} \)
3. Use a recursive formula
   \( A = \{x_i \mid x_i = x_{i-1} + 1, \ i=1, 2, 3, 4, 5, \ x_0 = 0\} \)
   \( i=1 \quad x_1 = x_0 + 1 = 0 + 1 = 1 \)
   \( i=2 \quad x_2 = x_1 + 1 = 1 + 1 = 2 \)
   \( i=3 \quad x_3 = x_2 + 1 = 2 + 1 = 3 \)
   \( i=4 \quad x_4 = x_3 + 1 = 3 + 1 = 4 \)
   \( i=5 \quad x_5 = x_4 + 1 = 4 + 1 = 5 \)

* Def. Set \( A \) is a subset of set \( B \), \( A \subseteq B \)
   if every element in \( A \) is an element in \( B \)
   Set \( A \) is a proper subset of \( B \), \( A \subset B \)
   if \( A \subseteq B \) and \( A \neq B \)

Venn Diagram

Weakness:
cannot distinguish
\( A \subseteq B \) from \( A \subset B \)

* elementwise technique of proof
Thm. $\emptyset \subseteq A$ for all sets $A$.

Proof by contradiction

1. Assume opposite (negation) of desired conclusion.
   Assume $\emptyset \not\subseteq A$ for some set $A$.

2. Get a contradiction:
   there is a $y \notin \emptyset$, $y \in A$
   contradiction

3. Therefore the assumption is false

Def. If $S$ is a set then $|S|$ is the cardinality of $S$ and is the number of elements in the set.

$A = \{a, b, \ldots, z\}$ $|A| = 26$ $|\emptyset| = 0$

Def. Two sets are equal, $A = B$ if and only if $A \subseteq B$ and $B \subseteq A$

Note: $\{1, 3, 5\} = \{3, 1, 5\}$  $\{1, 1, 3, 5, 5, 3\}$ order not important, only membership

Def. Given a set $S$, the power set of $S$, denoted by $P(S)$ is the set of all subsets of $S$

$P(S) = \{A \mid A \subseteq S\}$ has sets as elements

Ex. $S = \{a, b, c\}$

$P(S) = \{\emptyset, \{a\}, \{b\}, \{c\}, \{a, b\}, \{a, c\}, \{b, c\}, \{a, b, c\}\}$

$P(\emptyset) = \{\emptyset\}$

$P(\{a\}) = \{\emptyset, \{a\}\}$

$P(\{a, b\}) = \{\emptyset, \{a\}, \{b\}, \{a, b\}\}$

$|P(A)| = 2^{|A|}$

Guess $|P(A)| = |A|^3$

\[\begin{array}{c|c|c}
|A| & 1 & 2 \\
|P(A)| & 1 & 2 \\
\end{array}\]

$|P(A)| = 2^{|A|}$
Properties of Sets

Prove for all sets A, B, C:

(Use "elementwise approach"
To prove $A_1 \subseteq A_2$, take $x \in A_1$, prove $x \in A_2$)

1. $A \subseteq A$
   $x \in A$ $\Rightarrow$ $x \in A$

2. If $A \subseteq B$ and $B \subseteq C$ then $A \subseteq C$
   $x \in A$ $\Rightarrow$ $x \in B$ $\Rightarrow$ $x \in C$
   $A \subseteq B$
   $B \subseteq C$

3. If $A \subseteq B$ and $B \subseteq C$ then $A \subseteq C$
   (Same Venn Diagram as 2)
   a) Prove $A \subseteq C$:
      $x \in A$ $\Rightarrow$ $x \in B$ $\Rightarrow$ $x \in C$
      $A \subseteq B$
      $B \subseteq C$
   b) Prove $A \neq C$
      (Find a $y \in C$, with $y \notin A$)
      Since $B \subseteq C$, there is a $y \in C$, $y \notin B$
      Since $A \subseteq B$, $y \notin A$

4. If $A \subseteq B$ and $A \neq C$ then $B \neq C$
   (Find a $y \in B$, with $y \notin C$)
   Since $A \neq C$ there is a $y \in A$ with $y \notin C$
   Since $A \subseteq B$, $y \in B$

Def. the universal set, $U$, is the set of all objects under consideration.

Def. the set of no elements is called the empty set or null set and is denoted by $\emptyset$ or $\emptyset$.

*Note $\emptyset \neq \emptyset$, $\emptyset \notin \emptyset \subseteq \emptyset$ and is nonempty.
Cartesian Products

Def. The ordered \( n \)-tuple \((a_1, a_2, \ldots, a_n)\) is an ordered collection of elements where \(a_1\) is the first element, \(a_2\) is the second element, and \(a_n\) is the \(n^{th}\) element.

\((a_1, a_2, \ldots, a_n) = (b_1, b_2, \ldots, b_n)\) iff \(a_1 = b_1, a_2 = b_2, \ldots, a_n = b_n\)

Ordered 2 tuple = ordered pair

Def. Let \(A\) and \(B\) be sets. The cartesian product of \(A\) and \(B\) denoted by \(A \times B\) is the set of ordered pairs \((a, b)\) where \(a \in A\) and \(b \in B\).

\[A \times B = \{(a, b) | a \in A \text{ and } b \in B\}\]

Ex. Let \(A = \{1, 2, 3\}\) \(B = \{a, b, c\}\)

\[A \times B = \{(1, a), (1, b), (1, c), (2, a), (2, b), (2, c)\}\]

\[B \times A = \{(a, 1), (a, 2), (a, 3), (b, 1), (b, 2), (b, 3), (c, 1), (c, 2), (c, 3)\}\]

Note: \(|A| = 2 \quad |B| = 3 \quad |A \times B| = 2 \times 3 = 6\)

In general \(|A \times B| = |A| \times |B|\) for finite sets \(A, B\)

\[A \times B = B \times A\] unless \(A = \emptyset\) or \(B = \emptyset\) or \(|A| = |B|\)

Def. The cartesian product of the sets \(A_1, A_2, \ldots, A_n\) denoted by \(A_1 \times A_2 \times \cdots \times A_n\) is the set of ordered \(n\)-tuples:

\[A_1 \times A_2 \times \cdots \times A_n = \{(a_1, a_2, \ldots, a_n) | a_1 \in A_1, a_2 \in A_2, \ldots, a_n \in A_n\}\]

\[A_1 \times A_2 \times \cdots \times A_n = A^n \quad \text{n times}\]
Set Operations

Def. The union of two sets, \( A \) and \( B \), denoted \( A \cup B \)
\[
A \cup B = \{ x \mid x \in A \text{ or } x \in B \}
\]

Def. The intersection of two sets, \( A \) and \( B \), denoted \( A \cap B \)
\[
A \cap B = \{ x \mid x \in A \text{ and } x \in B \}
\]

Def. Two sets are disjoint if \( A \cap B = \emptyset \)

Thm. Principle of inclusion-exclusion
\[
|A \cup B| = |A| + |B| - |A \cap B|
\]

Def. The difference of \( A \) and \( B \), denoted \( A - B \)
\[
A - B = \{ x \mid x \in A \text{ and } x \notin B \}
\]

Def. The complement of \( A \), denoted \( \overline{A} \)
\[
\overline{A} = \{ x \mid x \in U \text{ and } x \notin A \}
\]

Thm. For all sets \( A \) and \( B \), \( A - B = A \cap \overline{B} \)
\[
x \in A - B \iff x \in A \text{ and } x \notin B
\]

\[x \in A - B \iff x \in A \cap \overline{B}\]
\[x \notin A \cup B \iff x \in A \cap \overline{B}\]

\[x \in A - B \iff x \in A \cap \overline{B}\]
Lemmas (small theorems)

For all sets $A$:

1. $A \cup \overline{A} = U$
2. $A \cap \overline{A} = \emptyset$
3. $A \cup \emptyset = A$
4. $A \cup \emptyset = A$
5. $A \cap \emptyset = \emptyset$

Lemma. For all sets $A, B$, if $A \subseteq B$ then $A \cap B = A$

\[
\begin{align*}
    x \in A \cap B & \iff x \in A \text{ and } x \in B \\
    & \iff x \in A
\end{align*}
\]

* Geometry style approach.

*Thm. Prove for all sets $A, B, C$
\[
\overline{A \cup (B \cap C)} = (\overline{C} \cup \overline{B}) \cap \overline{A}
\]

(may do elementwise)

\[
\begin{align*}
    & = \overline{A} \cap \overline{(B \cap C)} \\
    & = \overline{A} \cap \overline{B} \cap \overline{C} \quad \text{De Morgan's} \\
    & = \overline{A} \cap (\overline{B} \cup \overline{C}) \quad \text{De Morgan's} \\
    & = (B \cup C) \cap \overline{A} \quad \text{Comm prop } \cap \\
    & = (\overline{B} \cup \overline{C}) \cap \overline{A} \quad \text{Comm prop } \cup
\end{align*}
\]

*Thm. For all sets $A, B$ prove

\[
    A - (A - B) = A \cap B
\]

\[
\begin{align*}
    & = A \cap (A \cap \overline{B}) \quad \text{Thm } A - B = A \cap \overline{B} \\
    & = A \cap (\overline{A} \cup B) \quad \text{De Morgan's} \\
    & = A \cap (\overline{A} \cup B) \quad \text{Thm } \overline{A} = A \\
    & = (A \cap \overline{A}) \cup (A \cap B) \quad \text{Distributive} \\
    & = \emptyset \cup (A \cap B) \quad \text{Lemma } A \cap \overline{A} = \emptyset \\
    & = A \cap B \quad \text{Lemma } A \cup \emptyset = A
\end{align*}
\]
Thm. For all sets \( A \), \( \overline{\overline{A}} = A \)
\[
 x \in \overline{A} \iff x \notin A \iff x \in \overline{A} 
\implies \overline{A} \subseteq A 
\iff A \subseteq \overline{A}
\]

Thm. Set Identities

Idempotent
\( A \cup A = A \)
\( A \cap A = A \)

Commutative
\( A \cup B = B \cup A \)
\( A \cap B = B \cap A \)

Associative
\( A \cup (B \cup C) = (A \cup B) \cup C \)
\( A \cap (B \cap C) = (A \cap B) \cap C \)

Question: Does \( A \cup (B \cap C) = (A \cup B) \cap C \)?

Test with a Venn Diagram

\[
\begin{align*}
\text{AU(B \cap C)} & = \{2,3,5,6,7,35\} \text{ not equal} \\
(A \cup B) \cap C & = \{5,6,73\} \\
\text{Provides a counterexample}
\end{align*}
\]

Thm. Distributive Laws

\( A \cup (B \cap C) = (A \cup B) \cap (A \cup C) \)

Proof
\[
\begin{align*}
A \cup (B \cap C) & \iff x \in A \text{ and } x \in B \cap C \\
& \iff x \in A \text{ and } (x \in B \text{ and } x \in C) \\
& \iff x \in A \text{ and } (x \in B \text{ or } x \in C) \\
& \iff x \in (A \cup B) \cup (A \cup C)
\end{align*}
\]

Thm. De Morgan's Laws

\( (A \cup B) = \overline{\overline{A} \cap \overline{B}} \)

(\( A \cap B = \overline{\overline{A} \cup \overline{B}} \))

Proof
\[
\begin{align*}
x \in (A \cap B) & \iff x \in A \text{ and } x \in B \\
& \iff \neg x \in \overline{A} \text{ and } x \notin \overline{B} \\
& \iff x \in \overline{A} \text{ and } x \in \overline{B} \\
& \iff x \notin (A \cup B)
\end{align*}
\]

* Logic not \((a \text{ and } b)\) \iff not \(a\) or not \(b\)
not \((a \text{ or } b)\) \iff not \(a\) and not \(b\).
Def. Generalized Unions and Intersections

\[ A_1 \cap A_2 \cap \ldots \cap A_n = \bigcap_{i=1}^n A_i = \{ x \mid x \in A_1 \text{ and } x \in A_2 \text{ and } \ldots \text{ and } x \in A_n \} \]

\[ A_1 \cup A_2 \cup \ldots \cup A_n = \bigcup_{i=1}^n A_i = \{ x \mid x \in A_1 \text{ or } x \in A_2 \text{ or } \ldots \text{ or } x \in A_n \} \]

Ex. Let \( U = \mathbb{N} = \{ 1, 2, 3, \ldots \} \)
Let \( A_i = \{ i, i+1, i+2, \ldots \} \) for \( i = 1, 2, 3, \ldots \)
\[ A_1 = \{ 1, 2, 3, \ldots \} \]
\[ A_2 = \{ 2, 3, \ldots \} \]
\[ A_3 = \{ 3, 4, 5, \ldots \} \]
\[ \bigcup_{i=1}^n A_i = \{ 1, 2, 3, \ldots \} = \mathbb{N} \]
\[ \bigcap_{i=1}^n A_i = \{ n, n+1, \ldots \} = \mathbb{N} \]

Def. Computer Representation of Sets
(Must have a finite universe)

Make an arbitrary ordering of \( U = \{ a_1, a_2, \ldots, a_n \} \)

Def. A bit (binary digit) is a 1 or 0
A bit string is a finite sequence of bits
Represent a subset \( A \) of \( U \) with a bit string of length \( n \) where:
- The \( i \)th bit is 1 if \( a_i \in A \)
- The \( i \)th bit is 0 if \( a_i \notin A \)

Ex. \( U = \{ 1, 2, 3, \ldots, 10 \} \)
Let \( A = \{ 1, 2, 5, 8, 9 \} \)
\[ A = \{ 1001001100 \} \]
\[ \overline{A} = \{ 0011011001 \} \]

Def. The symmetric difference of \( A \) and \( B \), denoted \( A \oplus B \), is \( \{ x \mid x \in A \text{ or } x \in B \text{ and } x \notin (A \cap B) \} \)

\[ U \cap A \cap B \]
Thm. For all sets $A, B$

$$A \oplus B = (A \cup B) - (A \cap B)$$

*Proof.* $x \in A \oplus B \iff x \in A \land x \in B \land x \notin (A \cap B)$

$$\iff \cap x \in A \cup B \land x \notin (A \cap B)$$

$$\iff \cap x \in (A \cup B) - (A \cap B)$$

Thm. For all sets $A, B$

$$A \oplus B = (A - B) \cup (B - A)$$

*Proof.* $x \in A \oplus B \iff x \in A \land x \in B \land x \notin (A \cap B)$

$$\iff (x \in A \land x \notin B) \lor (x \in B \land x \notin A)$$

$$\iff x \in A - B \lor x \in B - A$$

$$\iff x \in (A - B) \cup (B - A)$$