Cardinality

Let's say $A$ and $B$ are finite sets.

If $|A| > |B|$, is there a function $f : A \rightarrow B$ which is 1:1?

- $A = \{1, 2, 3\}$
- $B = \{a, b\}$

$\begin{align*}
1 & \rightarrow a \quad \text{NO} \\
2 & \rightarrow b \quad \text{NO} \\
3 & \rightarrow ?
\end{align*}$

If $|A| < |B|$, is there a function $f : A \rightarrow B$ which is onto?

- $A = \{1, 2, 3\}$
- $B = \{a, b, c\}$

$\begin{align*}
1 & \rightarrow a \quad \text{NO} \\
2 & \rightarrow b \quad \text{NO} \\
3 & \rightarrow ?
\end{align*}$

So for finite sets $A$ and $B$, if there is a bijection $f : A \rightarrow B$ then $|A| = |B|$

Def: Define a relation, $\cong$, "isomorphism" on the set of all sets:

- $A \cong B$ if there is a bijection $f : A \rightarrow B$
- $(A$ is isomorphic to $B)$

Thm: Isomorphism relation $\cong$ is an equivalence relation on the set of all sets.

Reflexive: $\mathcal{I}_A : A \rightarrow A$ by $\mathcal{I}_A(x) = x$ is a bijection, so $A \cong A$ for all sets $A$.

Symmetric: If $A \cong B$ then $B \cong A$ for all sets $A, B$.

Since $A \cong B$ there is a bijection $f : A \rightarrow B$ then $f^{-1} : B \rightarrow A$ is a bijection. So $B \cong A$

Transitive: If $A \cong B$ and $B \cong C$ then $A \cong C$ for all sets $A, B, C$. 
Since \( A \cong B \) there is a bijection \( g: A \to B \)
Since \( B \cong C \) there is a bijection \( f: B \to C \)
We need to prove \( A \cong C \), we need to prove
\[ \text{fog}: A \to C \] is a bijection. To do so, we have:

**Thm:** If \( g: A \to B \) and \( f: B \to C \) are one-to-one functions then \( \text{fog}: A \to C \) is a one-to-one function.

**Proof:**
- \( g(a_1) \neq g(a_2) \) for \( a_1, a_2 \in A \)
- \( f(g(a_1)) \neq f(g(a_2)) \) (since \( g \) is one-to-one)
- \( \text{fog}(a_1) \neq \text{fog}(a_2) \) (Def \( \text{fog} \))

**Thm:** If \( g: A \to B \) and \( f: B \to C \) are onto functions then \( \text{fog}: A \to C \) is an onto function.

**Proof.** Take \( c \in C \). Since \( f \) is an onto function, there is a \( b \in B \) with \( f(b) = c \). Since \( g \) is an onto function there is an \( a \in A \) with \( g(a) = b \). So
\[ \text{fog}(a) = f(g(a)) = f(b) = c \]

Isomorphism is an equivalence relation on the set of all sets, and therefore the set of all sets can be partitioned into equivalence classes.
Def \( |\mathbb{P}| = \aleph_0 \) (aleph null) \( \mathbb{P} = \{1, 2, 3, \ldots\} \)

Thm. \( \mathbb{N} = \{0, 1, 2, 3\} \) \( |\mathbb{N}| = \aleph_0 \)

Proof: \( f: \mathbb{P} \to \mathbb{N} \) \( f(p) = p - 1 \) is a bijection

Thm. \( \mathbb{Z} = \{\ldots, -2, -1, 0, 1, 2, \ldots\} \) \( |\mathbb{Z}| = \aleph_0 \)

Proof: \( f: \mathbb{Z} \to \mathbb{P} \) \( f(z) = 2z + 1 \) if \( z \geq 0 \) is a bijection.
\( = -2z \) if \( z < 0 \)

\[ \begin{array}{cccccccc}
2 & \ldots & -2 & -1 & 0 & 1 & 2 & 3 & 4 & \ldots \\
\mathbb{P} & \begin{array}{c}
\xleftarrow{1} \\
\xrightarrow{2}
\end{array} & 3 & \xrightarrow{4} & 5 & \ldots
\end{array} \]

Thm. \( |\mathbb{Q}| = \aleph_0 \) \( \mathbb{Q} = \text{rationals} \)

Proof: Every rational number is in the above infinite matrix.
Follow the arrows and skip repetitions
\( P = \{1, 2, 3, 4, \ldots\} \) \( \mathbb{Q} = \{0, 1, \frac{1}{2}, \frac{1}{3}, \ldots\} \)

Proof: \( f: \mathbb{P} \to \mathbb{Q} \) is a bijection

Thm. \( |\mathbb{P} \times \mathbb{P}| = \aleph_0 \)

Proof: \( \begin{array}{cccccccc}
(1, 1) & (1, 2) & (1, 3) & \cdots \\
(2, 1) & (2, 2) & (2, 3) & \cdots \\
(3, 1) & (3, 2) & (3, 3) & \cdots
\end{array} \)

\( \mathbb{P} \times \mathbb{P} = \{1, 1, 1, 2, 1, 2, \ldots\} \) \( \mathbb{P} = \{1, 2, 3, 4, \ldots\} \)

Follow arrows \( f: \mathbb{P} \to \mathbb{P} \times \mathbb{P} \) is a bijection
Def. A countably infinite set is a set with cardinality \( \aleph_0 \).

A countable set is one which is either finite or countably infinite.

Def. An enumeration of \( A \) is a listing of \( A \):
\[ A = (a_1, a_2, \ldots, a_n, \ldots) \]
which lists \( A \) without repetition and has one \( a_n \) for each \( n \in \mathbb{N} \) (must be an infinite set - one for each \( n \in \mathbb{N} \)).

Thm. If a set can be enumerated, the set has cardinality \( \aleph_0 \).

Proof. \( A = (a_1, a_2, \ldots, a_n, \ldots) \) (definition of enumeration)
\[ f: \mathbb{N} \to A \] by \( f(n) = a_n \) is a bijection.

Thm. Subsets of countable sets are countable.

Proof. Prove subsets of \( \mathbb{N} \) are countable.

Let \( A \subseteq \mathbb{N} \).

If \( A \) is finite, it is countable.

Suppose \( A \) is infinite:
\[ A = (a_1, a_2, a_3, \ldots) \]
where \( a_1 = \text{least element} \), \( a_2 = \text{least element in } A - \{a_1, a_3, \ldots\} \), etc.

Thm. The countable union of countable sets is countable.

\[ \bigcup_{i \in I} A_i \]
Let \( I = \text{countable index set} \).
Either \( I = \{1, 2, \ldots, n\} \) or \( I = \mathbb{N} \).
a) Let \( I = \{1, 2, \ldots, n\} \)  
\[ a_{ij} = j^{th} \text{ element of set } A_i \]
\[ \bigcup_{i=1}^{n} A_i = \{a_{11}, a_{21}, \ldots, a_{1n}, a_{21}, a_{22}, \ldots, a_{2n}, \ldots\} \]

This is an enumeration of the union (if infinite) (each \( A_i \) is countable - assume at least one \( A_i \) is infinite).

b) Let \( I = \mathbb{P} \)
\[ \bigcup_{i=1}^{\infty} A_i \]
we will show it can be enumerated:
\[ A_1: a_{11} a_{12} a_{13} \ldots \]
\[ A_2: a_{21} a_{22} a_{23} \ldots \]
\[ A_n: a_{n1} a_{n2} a_{n3} \ldots \]
\[ \bigcup_{i=1}^{\infty} A_i \text{ can be enumerated} \]

( follow arrows and leave out repeats)

**Thm.** If \( S \) and \( T \) are countable then \( S \times T \)
is countable.
Proof: Supplement homework exercise.

**Ex.** \( |\mathbb{N} \times \mathbb{P}| = \aleph_0 \)  
\( \{1, 2, 3 \times \mathbb{Q}\} = \aleph_0 \)  
\( \mathbb{Z} \times \{0, \ldots, 23\} = \aleph_0 \)

**Def.** An infinite set which is not countable is called uncountable in finite.
Theorem: the real numbers are uncountably infinite (Cantor's Diagonal Argument).

Proof: Assume the real numbers are countably infinite and can be enumerated as
\[ \mathbb{R} = (r_1, r_2, \ldots, r_n, \ldots) \]

\[ r_1 = a_1 \cdot b_{11} b_{12} \ldots \]
\[ r_2 = a_2 \cdot b_{21} b_{22} \ldots \]
\[ r_3 = a_3 \cdot b_{31} b_{32} \ldots \]
\[ r_n = a_n \cdot b_{n1} b_{n2} \ldots \]

\[ a_i = \text{integer portion of } r_i \]
\[ b_{ij} = j^\text{th} \text{ decimal of } r_i \]
\[ b_{ij} \in \{0,1,2,\ldots,9\} \]

Define a real number \( \overline{b} = .\overline{b_{11} b_{22} \ldots b_{nn} \ldots} \)

where \( \overline{b_{i\cdot}} = \begin{cases} 0 & \text{if } b_{i\cdot} = 1,2,3,4,5,6,7,8,9 \\ 1 & \text{if } b_{i\cdot} = 0 \end{cases} \)

\( \overline{b} \) is a real number and should be on the list \( \mathbb{R} = (r_1, r_2, \ldots, r_n, \ldots) \)

but \( \overline{b} \neq r_n \) for \( n = 1,2,3,\ldots \) because \( b_{nn} \neq b_{nn} \)

(they are different at the \( n^\text{th} \text{ decimal place} \))

Contradiction: the real numbers cannot be enumerated and are uncountably infinite.

Note: Why does the above argument not work for the rational numbers?

\( \overline{5} = .\overline{55} \) is a real number, but we can't be sure it is a rational number.
Ex Prove \( [0,1) = \{ r | 0 \leq r < 1, \, r \in \mathbb{R}^2 \} \) is uncountably infinite:

Use same argument as with the real numbers

Change \( a_i = 0 \)

Ex Prove \( (-1,2) = \{ r | -1 < r < 2, \, r \in \mathbb{R}^2 \} \) is uncountably infinite:

Use same argument as with the real numbers

Change \( a_i = 1 \)

\( \overline{b} = 1. \overline{b_1 b_2 b_3 \ldots} \)

Ex. Prove the set of all sequences of a's, b's, c's is uncountably infinite:

Assume the set is countably infinite and can be enumerated: \( \mathcal{S} = (S_1, S_2, \ldots, S_n, \ldots) \)

\( S_1 = S_1(1), S_1(2), \ldots, S_1(n), \ldots \)

\( S_2 = S_2(1), S_2(2), \ldots, S_2(n), \ldots \)

\( \vdots \)

\( S_n = S_n(1), S_n(2), \ldots, S_n(n), \ldots \)

\( S_{i}(j) \) is \( j \)th value of sequence \( S_i \)

Define a sequence \( \overline{S} = S(1), S(2), \ldots, S(n), \ldots \)

where \( S(i) = \begin{cases} a & \text{if } S_i(i) = b \\ b & \text{if } S_i(i) = c \\ c & \text{if } S_i(i) = a \end{cases} \) \( i = 1, 2, 3, \ldots \)

\( \overline{S} \neq S_n \) for \( n = 1, 2, \ldots \) since \( S(n) \neq S_n(n) \)

(different at \( n \)th value)

Contradiction: the set of all sequences of a's, b's, c's cannot be enumerated and is uncountably infinite.
Ex: Prove the set of sequences of 0's & 1's is uncountably infinite.

Change previous argument: $S_i(j) \in \{0, 1\}$

$S(i) = \begin{cases} 
0 & \text{if } S_i(j) = 1 \\
1 & \text{if } S_i(j) = 0 
\end{cases}$ for $i = 1, 2, 3, \ldots$

Thm. If $A$ is uncountably infinite and $A \subseteq B$ then $B$ is uncountably infinite.

Proof: Contradiction. Assume $B$ is countable.

Since $A \subseteq B$ then $A$ is countable (subsets of countable sets are countable). Contradiction since $A$ is uncountably infinite.

Ex: Prove $\mathbb{R} \times \mathbb{Z}$ is uncountably infinite.

$\mathbb{R} \cong \mathbb{R} \times \{0\} \subseteq \mathbb{R} \times \mathbb{Z}$ if $A$ is uncountably infinite (isomorphic to $\mathbb{R}$) and $A \subseteq B$ then $B$ is uncountably infinite.

Thm. If $A$ is uncountably infinite and $B$ is countable then $A - B$ is uncountably infinite.

Proof: Contradiction. Assume $A - B$ is countable.

Then $(A - B) \cup B$ is countable (a countable union of countable sets is countable). But $A \subseteq (A - B) \cup B$ ($x \in A \Rightarrow (x \in A \text{ and } x \in B) \text{ or } x \in B$). So $A$ is countable. Since subset of countable sets are countable. Contradiction.