Integrals Involving Inverse Trig. Functions

By reversing certain differentiation formulas, we obtain the following integration formulas.

$$\int \frac{dx}{\sqrt{1-x^2}} = \sin^{-1} x + C, \int \frac{dx}{1+x^2} = \tan^{-1} x + C, \text{ and } \int \frac{dx}{|x|\sqrt{x^2-1}} = \sec^{-1} x + C.$$

These formulas can then be generalized to #16, 17, and 19 on p. TA-1 at the end of the book. We can drop the absolute value symbols.

Let's prove one of those.

Consider $\int \frac{dx}{a^2+x^2}$. Rewrite this as $\frac{1}{a^2}\int \frac{dx}{1+\left(\frac{x}{a}\right)^2}$. If we let $u = \frac{x}{a}$, then

$du = \frac{1}{a}dx$, or $adu = dx$. We get $\frac{1}{a^2}\int \frac{adu}{1+u^2}$ or $\frac{1}{a}\int \frac{1}{1+u^2}du = \frac{1}{a}\tan^{-1} u + C$.

Then switching back to x, it is $\frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right) + C$.

The other two formulas can be derived similarly.

Let's look at some examples:

a) $\int \frac{dx}{1+5x^2}$ Rewrite the integral as $\int \frac{dx}{1+(\sqrt{5}x)^2}$. Now let $u = \sqrt{5}x$.

Then $du = \sqrt{5}dx$. We obtain, $\frac{1}{\sqrt{5}}\int \frac{du}{1+u^2} = \frac{1}{\sqrt{5}}\tan^{-1} u + C =$

$$\frac{1}{\sqrt{5}}\tan^{-1}(\sqrt{5}x) + C.$$

b) $\int \frac{e^x}{\sqrt{1-e^{2x}}}dx = \int \frac{e^x}{\sqrt{1-(e^x)^2}}dx$. Let $u = e^x$. Then $du = e^x dx$.

We get $\int \frac{du}{\sqrt{1-u^2}} = \sin^{-1} u + C = \sin^{-1}(e^x) + C$.

Note: It is also true that $\int \frac{dx}{x\sqrt{x^2-1}} = \sec^{-1}|x| + C$.

c) $\int_{\sqrt{2}}^{2} \frac{dx}{x\sqrt{x^2-1}} = \sec^{-1}|x|\Big|_{\sqrt{2}}^{2} = \sec^{-1} 2 - \sec^{-1}\sqrt{2} = \frac{\pi}{3} - \frac{\pi}{4} = \frac{\pi}{12}$.

Exercises

1) $\int_{-1}^{1} \frac{dx}{1+x^2}$ Ans. $\frac{\pi}{2}$

2) $\int \frac{dx}{1+16x^2}$ Ans. $\frac{1}{4}\tan^{-1} 4x + C$

3) $\int \frac{dx}{\sqrt{9-4x^2}}$ Ans. $\frac{1}{2}\sin^{-1}(\frac{2x}{3}) + C$

4) $\int \frac{dx}{x\sqrt{5x^2-3}}$ Ans. $\frac{1}{\sqrt{3}}\sec^{-1}(\frac{\sqrt{5}x}{\sqrt{3}}) + C$

5) $\int \frac{dx}{x\sqrt{1-(\ln x)^2}}$ Ans. $\sin^{-1}(\ln x) + C$

6) Find the area of the region enclosed by the graphs of $\frac{1}{\sqrt{1-9x^2}}$,

y = 0, x = 0, and $x = \frac{1}{6}$. Ans. $\frac{\pi}{18}$

7) $\int_{-\sqrt{2}}^{\frac{-2}{\sqrt{3}}} \frac{dx}{x\sqrt{x^2-1}}$ Ans. $-\frac{\pi}{12}$

8) $\int_{1}^{3} \frac{dx}{\sqrt{x}(x+1)}$ Ans. $\frac{\pi}{6}$

9) $\int \frac{e^x}{4+e^{2x}}dx$ Ans. $\frac{1}{2}\tan^{-1}(\frac{e^x}{2}) + C$