INTERPOLATING MULTIPLICITY VARIETIES FOR $A_p^0(\mathbf{C}^n)$

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Abstract: A necessary and sufficient condition is obtained for a discrete multiplicity variety to be an interpolating variety for the space $A_p^0(\mathbf{C}^n)$.

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§1. Introduction. In this paper, we will consider interpolation problems for the space $A_p^0(\mathbf{C}^n)$, which is the ring of entire functions in $\mathbf{C}^n$ satisfying that for every $\epsilon > 0$, there exists a constant $A_\epsilon > 0$ such that $|f(z)| < A_\epsilon e^{\epsilon p(z)}$ for $z \in \mathbf{C}^n$, i.e., $\sup_{z\in\mathbf{C}^n}\{|f(z)|e^{-\epsilon p(z)}\} < +\infty$, where p is a weight (see § 2). The space $A_p^0(\mathbf{C}^n)$ is an important class of entire functions in both complex analysis and harmonic analysis. When $p(z) = |z|$, the ring $A_{|z|}^0(\mathbf{C}^n)$, via Fourier-Borel transformation, topologically isomorphic to the ring of infinite order differential operators. The space has a natural projective limit structure and under its locally convex topology it becomes a nuclear Fréchet algebra. This kind of algebra appears naturally in functional analysis.

Let $\{\zeta_k\}$ be a discrete set in $\mathbf{C}^n$ and $\{m_k\}$ a sequence of positive integers. If for any multi-indexed sequence $\{a_{k,I}\}_{k\in\mathbf{N},0\le|I|<m_k}$ of complex numbers in $A_p^0(V)$, the space of sequences $\{a_{k,I}\}_{k\in\mathbf{N},0\le|I|<m_k}$ satisfying that for any $\epsilon > 0$,

$$\sum_{|I|=0}^{m_k-1} |a_{k,I}| < A_\epsilon e^{\epsilon p(\zeta_k)}, \qquad k \in \mathbf{N},$$

for some constants $A_\epsilon > 0$, where $I = (i_1, \cdots, i_n) \in (\mathbf{Z}^+)^n$ and $|I| = i_1 + i_2 +$

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$\cdots + i_n$, there always exists an entire function f in $A_p^0(\mathbf{C}^n)$ such that

$$\frac{1}{I!}\frac{\partial^{|I|} f(\zeta_k)}{\partial z^I} = a_{k,I}, \quad \text{for} \quad k \in \mathbf{N}, 0 \le |I| < m_k. \tag{1.1}$$

We will then say that $V := \{(\zeta_k, m_k)\}$ is an interpolating variety for $A_p^0(\mathbf{C}^n)$. Clearly the condition (1.1) means that f has a prescribed finite collection of Taylor coefficients at each ζ_k. If $m_k = 1$ for all k, then (1.1) simply means that f has prescribed values at given points ζ_k.

Note that the constant A_ϵ in the definition of $A_p^0(\mathbf{C}^n)$ depends on arbitrarily given ϵ, which makes it impossible to become a space $A_p(\mathbf{C}^n)$ for any weight p, where $A_p(\mathbf{C}^n)$ is the algebra of entire functions in $\mathbf{C}^n$ satisfying that $|f(z)| < Ae^{Bp(z)}, z \in \mathbf{C}^n$ for some fixed $A, B > 0$. To study problems such as analytic continuation for Dirichlet series and representation of analytic solutions of partial differential equations of infinite order, one needs to consider the interpolation problem for the space $A_p^0(\mathbf{C}^n)$ (see e.g. [BG][BKS][BLV] and references therein), instead of the one for $A_p(\mathbf{C}^n)$, for which various results for interpolation are known. Note that the growth condition for $A_p^0(\mathbf{C}^n)$ is evidently more restrictive than the one for $A_p(\mathbf{C}^n)$, which makes the interpolation problem for $A_p^0(\mathbf{C}^n)$ more delicate than the one for $A_p(\mathbf{C}^n)$. In [BKS], a sufficient interpolation condition for $A_p^0(\mathbf{C}^n)$, using distribution of points of V in a "tube neighborhood" of V, was obtained from $A_p(\mathbf{C}^n)$ for certain varieties by expressing $A_p^0(\mathbf{C}^n)$ as a sort of inductive limit of $A_p(\mathbf{C}^n)$ (cf. § 2). It however does not provide methods for necessary interpolation conditions. The main purpose of this paper is to give a similar interpolation condition, which is both necessary and sufficient for interpolation in $A_p^0(\mathbf{C}^n)$ and which applies to arbitrary discrete multiplicity varieties in $\mathbf{C}^n$. We will state the theorem in §2 and give its proof in §3.

§2. **Definitions and Results**. We first fix some notions and notations, which will be used throughout the paper.

Definition 2.1. A plurisubharmonic function $p:\mathbf{C}^n \to [0,\infty)$ is called a weight (function) if it satisfies the following conditions:

$$\log(1+|z|^2) = o\{p(z)\} \tag{2.1}$$

and

$$p(z) = p(|z|), \quad p(2z) = O\{p(z)\}. \tag{2.2}$$

Definition 2.2 . Let $A(\mathbf{C}^n)$ be the ring of all entire functions on $\mathbf{C}^n$. Then

$A_p^0(\mathbf{C}^n) =$

$\{f \in A(\mathbf{C}^n) : \forall\epsilon > 0, \exists A_\epsilon > 0, \text{ such that } |f(z)| \le A_\epsilon \exp(\epsilon p(z)), z \in \mathbf{C}^n\}.$

A simple but important example of weighted spaces $A_p^0(\mathbf{C}^n)$ is $A_{|z|}^0(\mathbf{C}^n)$, which is the space of entire functions of infraexponential type and plays important roles in Dirichlet series and Fabry type gap theorems.

Let $f \not\equiv 0$ be a holomorphic function on an open connected neighborhood of $\zeta \in \mathbf{C}^n$. Then a series $f(z) = \sum_{j=\nu}^{\infty} \mathcal{P}_j(z-\zeta)$ converges uniformly on some neighborhood of ζ and represents f on this neighborhood. Here $\mathcal{P}_j$ is a homogeneous polynomial of degree j and $\mathcal{P}_\nu \not\equiv 0$. The nonnegative integer ν, uniquely determined by f and ζ, is called the zero multiplicity, or zero divisor of f at ζ, denoted by $\mathrm{div}_f(\zeta)$.

Let $V = \{(\zeta_k, m_k)\}$ be a multiplicity variety; that is, a discrete set $\{\zeta_k\} \subset \mathbf{C}^n$ with $|\zeta_k| \to \infty$ together with a sequence $\{m_k\}$ of positive integers. We write $V \subseteq f^{-1}(0)$ if $\mathrm{div}_f(\zeta_k) \ge m_k$ for each k, i.e., each ζ_k is a zero of f of multiplicity at least m_k. Associated to V, there is a unique closed ideal in $A(\mathbf{C}^n)$,

$$J = J(V) := \{f \in A(\mathbf{C}^n) : \text{div}_f(\zeta_k) \geq m_k, \forall k\}.$$

Two entire functions g, h in $\mathbf{C}^n$ can be identified modulo J if and only

$$\frac{\partial^{|I|} g(\zeta_k)}{\partial z^I} = \frac{\partial^{|I|} h(\zeta_k)}{\partial z^I}, 0 \leq |I| < m_k, k \in \mathbf{N},$$

here and throughout the paper, we use I to denote a muti-index; that is, $I = (i_1, \cdots, i_n) \in (\mathbf{Z}^+)^n, \mathbf{Z}^+ = \{0, 1, 2, \cdots\}$. The quotient space $A(\mathbf{C}^n)/J$ can be identified to the space $A(V)$ of all sequences $\{a_{k,I}\}_{k \in \mathbf{N}, 0 \leq |I| < m_k}$ of complex numbers, which can be described as "analytic functions " on V. The map

$$\rho : \rho(f) = \{\frac{\partial^{|I|} f(\zeta_k)}{I! \partial z^I}\}_{k \in \mathbf{N}, 0 \leq |I| < m_k} \tag{2.3}$$

is the natural restriction map from $A(\mathbf{C}^n)$ into $A(V)$.

Definition 2.3. Let $V = \{(\zeta_k, m_k)\}$ be a multiplicity variety on $\mathbf{C}^n$. Then we define

$$A_p^0(V) = \{a = \{a_{k,I}\} \in A(V) : \forall \epsilon > 0, \exists A_\epsilon > 0,$$
$$\text{such that} \sum_{|I|=0}^{m_k - 1} |a_{k,I}| \leq A_\epsilon \exp(\epsilon p(\zeta_k)), k \in \mathbf{N}\}.$$

Using Cauchy's estimates, it is easy to check that ρ is a map from $A_p^0(\mathbf{C}^n)$ to $A_p^0(V)$. But, in general, the space $A_p^0(V)$ is too large.

Definition 2.4. A multiplicity variety $V = \{(\zeta_k, m_k)\}$ is an interpolating variety for $A_p^0(\mathbf{C}^n)$ if the restriction map ρ is surjective from $A_p^0(\mathbf{C}^n)$ to $A_p^0(V)$.

Clearly, that V is an interpolating variety for $A_p^0(\mathbf{C}^n)$ means that for any multi-indexed sequence $\{a_{k,I}\} \in A_p^0(V)$ there exists an entire function $f \in A_p^0(\mathbf{C}^n)$ such that $\frac{\partial^{|I|} f(\zeta_k)}{I! \partial z^I} = a_{k,I}$ for all $k \in \mathbf{N}$ and $0 \leq |I| < m_k$; i.e., f has a described finite collection of Taylor coefficients (and f has prescribed values if $m_k = 1$ for all k).

We obtain the following both necessary and sufficient interpolation conditions, which applies to arbitrary multiplicity varieties in $\mathbf{C}^n$.

Theorem 2.5 *Let $V = \{(\zeta_k, m_k)\}$ be a multiplicity variety in $\mathbf{C}^n$ and $N \geq n$ a positive integer. Then V is an interpolating variety for $A_p(\mathbf{C}^n)$ if and only if there exist an entire holomorphic mapping $f = (f_1, f_2, \cdots, f_N)$ with $f_j \in A_p^0(\mathbf{C}^n)$ and a positive function $q(z) = o\{p(z)\}$ such that $V \subseteq f^{-1}(0)$ and, for some constants $\epsilon, C > 0$, each connected component of the set*

$$S_q(f; \epsilon, C) := \{z \in \mathbf{C}^n : |f(z)| < \epsilon e^{-Cq(z)}\} \tag{2.4}$$

contains at most one point in V and such a component has diameter at most one.

Remark. (i) The above condition is given by means of distribution of points of V in a " tube neighborhood" $S(f; \epsilon, c)$ of the variety V. Distribution of points of V in such a tube neighborhood of V plays important roles in study of interpolation problems and slowly decreasing ideals (see e.g. [BKS][BT][LV]). A similar sufficient condition was given in [BKS, Theorem 3.2] for the case when V is the complete intersection of zero sets of some locally slowly decreasing functions in $A_p^0(\mathbf{C}^n)$, which is the main interpolation theorem in [BKS] used to prove the gap theorems in [BKS, § 4]. The notion of $f_1, f_2, \cdots, f_n$ being locally slowly decreasing is that there are positive constants ϵ, C, C_1, C_2 and a weight q such that the set $S_q(f; \epsilon, C)$ has bounded connected components, which are such that $p(z) \leq C_1 p(\zeta) + C_2$ if z, ζ belong to the same component. It seems that the condition on the diameter of connected components of the "tube" is missing in the theorem in [BKS], which can not be deduced from the "locally slowly decreasing" assumption due to the presence of multiplicities.

(ii) Note that the multiplicity varieties in Theorem 2.5 may be arbitrarily given. If $m_k = 1$ for all k, then the condition in Theorem 2.5 is equivalent to an estimate on minors of the Jacobian matrix of the entire holomorphic mapping f in the theorem. Some other related work may be found in [BLV], [BKS], [BT], [Li], and [LV]. These work inspired and benefited the present paper.

We conclude this section by the following corollary, as an illustration for use of the both necessary and sufficient conditions of Theorem 2.5. It does not seem trivial to see whether an interpolating variety for $A^0_{|z|}(\mathbf{C}^n)$ is an interpolating variety for $A_{e^{|z|}}(\mathbf{C}^n)$. This is however a trivial consequence of the following general result.

Corollary 2.6. *If a multiplicity variety $V = \{(\zeta_k, m_k)\}$ in $\mathbf{C}^n$ is an interpolating variety for $A^0_p(\mathbf{C}^n)$, then it is also an interpolating variety for $A^0_u(\mathbf{C}^n)$ for any weight $u \geq p$.*

Proof. By the necessary condition of Theorem 2.5, there exist an entire holomorphic mapping $f = (f_1, f_2, \cdots, f_n)$ and a positive function $q(z) = o\{p(z)\}$ satisfying the conditions in Theorem 2.5. Since $u \geq p$, we have that $A^0_p(\mathbf{C}^n) \subseteq A^0_u(\mathbf{C}^n)$ and $q(z) = o\{u(z)\}$. Thus, all the conditions in Theorem 2.5 also hold for $A^0_u(\mathbf{C}^n)$. By the sufficient condition of Theorem 2.5, V is an interpolating variety for $A^0_u(\mathbf{C}^n)$.

§3. **Proof of Theorem 2.1.** For convenience, in the following proof we will use $0 < \epsilon < 1, c > 0$ to denote numerical constants, which may dependent on n and the actual value of which may vary from one occurences to the next.

We first give the proof of the necessity, which is rather complicated. To this end, we first write down explicitly the projective limit topologies of $A^0_p(\mathbf{C}^n)$ and $A^0_p(V)$ by specifying their neighborhood bases. For each positive integer m, we introduce the following space $A_m = \{f \in A(\mathbf{C}^n) : \|f\|_{m,\infty} < +\infty\}$, where $\|f\|_{m,\infty} := \sup_{z\in\mathbf{C}^n}\{|f(z)|e^{-\frac{1}{m}p(z)}\} < \infty\}$. The space $A_p^0(\mathbf{C}^n)$ can be identified with the space $\{(f, f, f, \cdots,) : f \in A_p^0(\mathbf{C}^n)\}$, which is a subspace of the product space $A_1 \times A_2 \times A_3 \times \cdots$ and is the projective limit of the family $\{A_m\}_{m=1}^{\infty}$ with respect to the natural projections of A_m to A_l for $l < m$ (see e.g. [S] for basics of projective limit topology). A neighborhood base of $f \in A_p^0(\mathbf{C}^n)$ is given by all the intersections

$$A_p^0(\mathbf{C}^n) \cap (\cap_{m\in H} U_m) \qquad (3.1),$$

where U_m is any neighborhood of f in A_m with respect to the topology induced by $||f||_{m,\infty}$ and H is any finite subset of $\mathbf{N}$. $A_p^0(\mathbf{C}^n)$ is metrizable as (by the above identification) a subset of a product of countable family of metrizable topological vector spaces and complete as a projective limit of complete locally convex spaces.

In the same way we set $A_m(V) = \{a = \{a_{k,I}\} : \|a\|_{m,\infty} < +\infty$, where $\|a\|_{m,\infty} := \sup_{k\in\mathbf{N},|I|<m_k}\{\sum_{|I|=0}^{m_k-1} |a_{k,I}|e^{-\frac{1}{m}p(\zeta_k)}\}$. Then $A_p^0(V)$ can be identified with the space $A(V) = \{(a, a, \ldots, a, \ldots) : a \in A_p^0(V)\}$, which is a subspace of $A_1(V) \times A_2(V) \times A_3(V) \times \cdots$ and is the projective limit of $A_m(V)$ with respect to the natural projections of $A_m(V)$ to $A_l(V)$ for $l < m$. A neighborhood base of $a \in A_p^0(V)$ is given by all the intersections

$$A_p^0(V) \cap (\cap_{m\in I} V_m), \qquad (3.2)$$

where V_m is any neighborhood of a with respect to the topology in $A_m(V)$ induced by $||a||_{m,\infty}$, where I is any finite subset of $\mathbf{N}$. The space $A_p^0(V)$ is also metrizable and complete.

Consider the map $\varphi : A_p^0(\mathbf{C}^n) \mapsto A_p^0(V)$ defined in (2.3). Then it is surjective since V is an interpolating variety for $A_p^0(\mathbf{C}^n)$. It is also easy to check that φ is linear and continuous. Thus, by the open mapping theorem (see e.g. [H, p.294]), φ maps every neighborhood of 0 in $A_p^0(\mathbf{C}^n)$ onto a neighborhood of 0 in $A_p^0(V)$.

We set for each positive integer m,

$$U_m^0 = \{f \in A_p^0(\mathbf{C}^n) : \|f\|_{m,\infty} < L_m\},$$

where $L_m > 1$ is a (yet to be determined) positive number. We claim that we can take L_m properly so that the image $\varphi(\cap_{j=1}^m U_j^0$ contains a set of the form

$$W_m^0 := \{a = \{a_{k,I}\} \in A_p^0(V) : ||a||_{l_m,\infty} \leq \gamma_m\}, \tag{3.3}$$

where l_m, γ_m are positive numbers, and γ_m satisfies the condition that $\gamma_m \geq 1$. In fact, it is clear that $U_m^0 = U_m \cap A_p^0(\mathbf{C}^n)$, where $U_m = \{f \in A(\mathbf{C}^n) : \|f\|_{m,\infty} < L_m\}$. By (3.1), we know that U_m^0 is a neighborhood of 0 in $A_p^0(\mathbf{C}^n)$. Then $\cap_{j=1}^m U_j^0$ is also a neighborhood of 0 in $A_p^0(\mathbf{C}^n)$. Since the restriction map φ maps a neighborhood of 0 in $A_p^0(\mathbf{C}^n)$ onto a neighborhood of 0 in $A_p^0(V)$, $\varphi(\cap_{j=1}^m U_j^0)$ contains a neighborhood of 0 in $A_p^0(V)$ and so, by (3.2), contains an open set of the form $(\cap_{m \in I} V_m) \cap A_p^0(V)$, where V_m is a neighborhood of 0 with respect to the topology in $A_m(V)$ and I is a finite subset of $\mathbf{N}$. We then deduce that there exist an integer $l_m > 0$ and a $\gamma_m > 0$ such that $\varphi(\cap_{j=1}^m U_j^0)$ contains the set W_m^0 in (3.3). However, the positive constant γ_m obtained above might not satisfy the required condition that $\gamma_m \geq 1$. If this happens for some m, we then need to revise the above sets. Suppose that m is the smallest positive integer so that $\gamma_m < 1$ (m might be 1). We then replace L_j by $\frac{1}{\gamma_m} l_j$ for $1 \leq j \leq m$, and replace U_j^0 by

$$\begin{aligned}\hat{U}_j^0 &:= \frac{1}{\gamma_m} U_j^0 := \{\frac{1}{\gamma_m} f : f \in U_j^0\} \\ &= \{f \in A_p^0(\mathbf{C}^n) : \|f\|_{j,\infty} < \frac{1}{\gamma_m} L_j\}\end{aligned}$$

for $1 \leq j \leq m$. One can then check, in view of the lineality of φ, that for each $1 \leq i \leq m$,

$$
\begin{aligned}
&\varphi(\cap_{j=1}^{i} \hat{U}_j^0) = \varphi(\cap_{j=1}^{i}(\frac{1}{\gamma_m} U_j^0)) \\
&\supseteq \frac{1}{\gamma_m}\varphi(\cap_{j=1}^{i} U_j^0) \\
&\supseteq \frac{1}{\gamma_m} W_i^0 = \{a = \{a_{k,I}\} \in A_p^0(V) : ||a||_{l_i,\infty} \leq \frac{\gamma_i}{\gamma_m} := \hat{\gamma}_i\} := \hat{W}_i^0,
\end{aligned}
$$

where $\hat{\gamma}_i = \frac{\gamma_i}{\gamma_m} \geq 1$ for each $1 \leq i \leq m$, in view of the fact that m is the smallest integer satisfying that $\gamma_m < 1$ and thus that $\hat{\gamma}_i = \frac{\gamma_i}{\gamma_m} \geq \gamma_i \geq 1$ for $1 \leq i \leq m-1$, and also that $\hat{\gamma}_m = \frac{\gamma_m}{\gamma_m} = 1$. Thus, we can replace W_i^0 by $\hat{W}_i^0$ for each $1 \leq i \leq m$, which satisfies the desired requirement that $\hat{\gamma}_i \geq 1$ for each $1 \leq i \leq m$. We can continue this process. If $\gamma_{m+1} \geq 1$, we have nothing to revise. If $\gamma_{m+1} < 1$, we then use the above way to get revised sets $\hat{W}_i^0, 1 \leq i \leq m+1$, for which we have that $\hat{\gamma}_i \geq 1$ for $1 \leq i \leq m+1$. Continuing this way, we eventually obtain a sequence of sets in $A_p^0(\mathbf{C}^n)$, still denoted by U_m^0, and a sequence of sets in $A_p^0(V)$, still denoted by W_m^0, which satisfy that $\varphi(\cap_{j=1}^{m} U_j^0) \supseteq W_m^0$ and $\gamma_m \geq 1$ for each integer $m \geq 1$. This proves the claim.

Next, we will use W_m^0 to produce a sequence of functions with certain "good" properties, which will help us to construct the desired mapping in the theorem. For each fixed $k \in \mathbf{N}$ and $1 \leq i \leq n$, the fact that the set $\varphi(\cap_{j=1}^{m} U_j^0)$ contains W_m^0 in (3.3) implies that there exists a sequence $\{g_{i,k,m}\}_{m=1}^{\infty}$ of entire functions such that $g_{i,k,m} \in \cap_{j=1}^{m} U_j^0$ and $\varphi(g_{i,k,m}) = \{\frac{\partial^{|I|} g_{i,k,m}(\zeta_l)}{I! \partial z^I}\}_{k \in \mathbf{N}, 0 \leq |I| < m_k} \in W_m^0$ with all the terms in this sequence being zero except one being 1, specified as follows

$$
\frac{\partial^{|I|} g_{i,k,m}(\zeta_l)}{\partial I! z^I} = 0, \forall l, \forall 0 \leq |I| < m_l; \quad \text{except that} \quad \frac{\partial^{l_k} g_{i,k,m}(\zeta_k)}{l_k! \partial z_i^{l_k}} = 1, \quad (3.4)
$$

where $l_k = \frac{m_k}{2}$ if m_k is even and $l_k = \frac{m_k - 1}{2}$ if m_k is odd. (This sequence clearly belongs to W_m^0. And it is here where we used the fact that $\gamma_m \geq 1$). The fact that $g_{i,k,m} \in \cap_{j=1}^m U_j^0$ implies that $\|g_{i,k,m}\|_{j,\infty} < L_j$ for $1 \leq j \leq m$ and so that

$$|g_{i,k,m}(z)| < L_j e^{\frac{p(z)}{j}}, 1 \leq j \leq m, z \in \mathbf{C}^n. \tag{3.5}$$

In particular, $|g_{i,k,m}(z)| \leq L_1 e^{p(z)}, z \in \mathbf{C}^n$. By (2.2), it is easy to check that there are two constants $A, B > 1$ such that

$$p(w) \leq Ap(z) + B \tag{3.6}$$

whenever $|w - z| < 2\sqrt{n}$. Thus, we see that $\{g_{i,k,m}\}_{m=1}^\infty$ is uniformly bounded in compact sets in $\mathbf{C}^n$ and so that $\{g_{i,k,m}\}$ is a normal family by Montel's theorem (see e.g. [G]). Thus, by passing to a subsequence we can assume that $\{g_{i,k,m}\}$ converges to a function $g_{i,k}$ in $\mathbf{C}^n$ as $m \to \infty$, which is an entire function in $\mathbf{C}^n$ by the Weierstrass theorem. Clearly, $g_{i,k}$ also satisfies that

$$\frac{\partial^{|I|} g_{i,k}(\zeta_l)}{\partial z^I} = 0, \forall l, \forall 0 \leq |I| < m_l; \quad \text{except that} \quad \frac{\partial^{l_k} g_{i,k}(\zeta_k)}{l_k! \partial z_i^{l_k}} = 1, \tag{3.7}$$

since each $g_{i,k,m}$ satisfies (3.4). Also, by (3.5) and noting that $\lim_{m\to\infty} g_{i,k,m}(z) = g_{i,k}(z)$, for each m and each $z \in \mathbf{C}^n$ there exists an integer $m_0 > m$ such that

$$\begin{aligned} |g_{i,k}(z)| &\leq |g_{i,k}(z) - g_{k,i,m_0}(z)| + |g_{k,i,m_0}(z)| \\ &\leq 1 + |g_{k,i,m_0}(z)| \leq 1 + L_j e^{\frac{p(z)}{j}} \end{aligned}$$

for each $1 \leq j \leq m_0$. In particular,

$$|g_{i,k}(z)| \leq 1 + L_m e^{\frac{p(z)}{m}} \leq 2L_m e^{\frac{p(z)}{m}}.$$

Since this inequality is true for each m, we deduce that

$$|g_{i,k}(z)| \leq \exp(\inf_m \{\log(2L_m) + \frac{1}{m} p(z)\}) := \exp(q_1(z)), \tag{3.8}$$

where

$$q_1(z) = \inf_m \{\log(2L_m) + \frac{1}{m} p(z)\}. \tag{3.9}$$

Clearly, q is a small function of p, i.e., $q_1(z) = o\{p(z)\}$.

Fix a positive number K satisfying that

$$\int_{\mathbf{C}^n} \frac{1}{(1+|z|)^K} d\sigma := L < +\infty, \tag{3.10}$$

where $d\sigma$ is the Euclidean volume element in $\mathbf{C}^n$. Now we define for each fixed integer $i (1 \le i \le n)$,

$$f_i(z) = \sum_{k=1}^{\infty} h_{i,k}(z) \frac{1}{(1+|\zeta_k|)^{K+1}} \exp(-2nAq_1(\zeta_k)), z \in \mathbf{C}^n \tag{3.11}$$

where $h_{i,k} = g_{i,k}^2$ if m_k is evan and $h_{i,k} = (z_i - \zeta_{k,i}) g_{i,k}^2$ if m_k is odd; $z = (z_1, z_2, \cdots, z_n)$, $\zeta_k = (\zeta_{k,1}, \cdots, \zeta_{k,n})$, and A is the number in (3.6). We will prove that $f_i \in A_p^0(\mathbf{C}^n)$. We denote by $f_{i,k}$ the general term of the series in (3.11). We then have, by virtue of (3.8), that

$$\begin{aligned} |f_{i,k}(z)| &\le (|z| + |\zeta_k|) e^{2q_1(z)} \frac{1}{(1+|\zeta_k|)^{K+1}} \exp(-2nAq_1(\zeta_k)) \\ &\le (1+|z|)(1+|\zeta_k|) e^{2q_1(z)} \frac{1}{(1+|\zeta_k|)^{K+1}} \exp\big(-2nAq_1(\zeta_k)\big) \\ &= (1+|z|) e^{2q_1(z)} \frac{1}{(1+|\zeta_k|)^K} \exp\big(-2nAq_1(\zeta_k)\big). \end{aligned} \tag{3.12}$$

Set $d_k = \min\{1, \inf_{l \ne k}\{|z_l - \zeta_k|\}\}$, and $\mathcal{D}_k = B(\zeta_k, \frac{d_k}{2})$, the ball centered at ζ_k with radius $\frac{d_k}{2}$. Then $d_k \le 1$ and $\mathcal{D}_k \cap \mathcal{D}_l = \emptyset$ for $k \ne l$. By (3.9) and (3.6), when $|z - w| \le 2(\sqrt{n} + 1)$, we have that

$$\begin{aligned} q_1(z) &\le \inf_m \{\log(2L_m) + \frac{1}{m}(Ap(w) + B)\} \\ &\le A \inf_m \{\log(2L_m) + \frac{1}{m} p(w)\} + B = Aq(w) + B, \end{aligned} \tag{3.13}$$

where A and B are the numbers in (3.6). If $d_k < 1$, then there is a $z_j \in V \cap \mathcal{B}(\zeta_k, 1)$ such that $z_j \neq \zeta_k$ and $d_k = |z_j - \zeta_k|$. Recall the $\mathbf{C}^n$ version of Schwarz's Lemma (see e.g. [Gu,p7]): If f is holomorphic in an open neighborhood of a closed ball $\bar{B}(\zeta, r)$ in $\mathbf{C}^n$ centered at ζ and with radius r, $|f(z)| \leq M$ for $z \in B(\zeta, r)$, and $\frac{\partial^{|I|} f}{\partial z^I}(\zeta) = 0$ whenever $|I| < m$ for some $m \in \mathbf{N}$, then $|f(z)| \leq M r^{-m} |z - \zeta|^m$ for $z \in \bar{B}(\zeta, r)$. We will apply this result to the function $\frac{\partial^{l_j} g_{i,j}(z)}{\partial z_i^{l_j}}$ in the disk $B(\zeta_k, 1)$. By (3.7) we know that $\frac{\partial^{l_j} g_{i,j}(\zeta_j)}{\partial l_j ! \partial z_i^{l_j}} = 1$ and $\frac{\partial^{l_j} g_{i,j}(\zeta_k)}{l_j ! \partial z_i^{l_j}} = 0$. Also, by Cauchy's estimate, we know that

$$\frac{\partial^{l_j} g_{i,j}(z)}{l_j ! \partial z_i^{l_j}} \leq c \max_{w \in \mathbf{C}^n : |w - z| \leq 1} \{|g_{i,j}(w)|\} \leq c e^{A q_1(\zeta_k) + B},$$

in view of (3.8) and (3.13). Thus, by the Schwarz Lemma,

$$\left|\frac{\partial^{l_j} g_{i,j}(z)}{\partial z_i^{l_j}}\right| \leq c e^{A q_1(\zeta_k) + B} |z - \zeta_k|$$

for $|z - \zeta_k| < 1$, and in particular,

$$1 = \left|\frac{\partial^{l_j} g_{i,j}(\zeta_j)}{\partial z_i^{l_j}}\right| \leq c e^{A q_1(\zeta_k) + B} |\zeta_j - \zeta_k|,$$

or $d_k = |\zeta_j - \zeta_k| \geq \epsilon e^{-A q_1(\zeta_k)}$. This inequality is obviously also true if $d_k = 1$. Therefore in any case the volume of the ball $\mathcal{D}_k$ satisfies that

$$vol \mathcal{D}_k = \frac{\pi^n}{n!} (\frac{d_k}{2})^{2n} \geq \epsilon e^{-2n A q_1(\zeta_k)}.$$

We thus deduce, by (3.12), that

$$\begin{aligned} |f_{i,k}(z)| &\leq (1 + |z|) e^{2 q_1(z)} \frac{1}{vol(\mathcal{D}_k)} \int_{\mathcal{D}_k} \frac{1}{(1 + |\zeta_k|)^K} \exp\big(-2n A q_1(\zeta_k)\big) d\sigma \\ &\leq c (1 + |z|) e^{2 q_1(z)} \int_{\mathcal{D}_k} \frac{1}{(1 + |\zeta_k|)^K} d\sigma, \end{aligned} \tag{3.14}$$

where $d\sigma$ is the Euclidean volume element in $\mathbf{C}^n$. Note that if $z \in \mathcal{D}_k$,

$$1 + |z| < 1 + |z - \zeta_k| + |\zeta_k| < 2 + |\zeta_k| < 2(1 + |\zeta_k|).$$

Therefore, in view of the fact that $\mathcal{D}_k \cap \mathcal{D}_l = \emptyset$ for $k \neq l$, we have that

$$\begin{aligned}\sum_{k=1}^{\infty} \int_{\mathcal{D}_k} \frac{1}{(1+|\zeta_k|)^K} d\sigma &\leq \sum_{k=1}^{\infty} \int_{\mathcal{D}_k} \frac{2^K}{(1+|z|)^K} d\sigma \\ \leq 2^K \int_{\mathbf{C}^n} \frac{1}{(1+|z|)^K} d\sigma &= 2^N L\end{aligned} \tag{3.15}$$

by (3.10). Also, by (3.13), $q(w) \leq Aq_1(z) + B$ whenever $|w - z| < 1$. We thus have showed that the series $f_i = \sum_{k=1}^{\infty} f_{i,k}$ converges uniformly in compact sets in $\mathbf{C}^n$ and so that f_i is an entire function in $\mathbf{C}^n$. Moreover, by virtue of (3.14) and (3.15), we have that

$$|f_i(z)| \leq c(1 + |z|)e^{2q_1(z)}. \tag{3.16}$$

But $(1 + |z|)e^{2q_1(z)} = e^{\log(1+|z|)+2q_1(z)} = e^{o\{p(z)\}}$ by (2.1) and (3.9). We thus conclude that $f_i \in A_p^0(\mathbf{C}^n)$.

Let $f = (f_1, f_2, \cdots, f_n)$. It is obvious that $V \subseteq f^{-1}(0)$ by the construction of each f_i (see (3.7) and (3.11)). Next we will find a positive function such that a tube neighbourhood $S(f; \epsilon, C)$ satisfies the conditions in the theorem. By (3.11) and (3.7) one can check that $f_i, 1 \leq i \leq n$, can be expanded into the following power series at each ζ_k,

$$\begin{aligned}f_i(z) &= c_k(z_i - \zeta_{k,i})^{m_k} + \\ &+ \sum_{i_1+\cdots+i_n \geq m_k+n_k}^{\infty} C_{i_1,\cdots,i_n}(\zeta_1 - \zeta_{k,1})^{i_1} \cdots (z_i - \zeta_{k,i})^{i_j} \cdots (\zeta_n - \zeta_{k,n})^{i_n},\end{aligned} \tag{3.17}$$

where

$$c_k = \frac{1}{(1 + |\zeta_k|)K + 1} \exp\bigl(-2nAq_1(\zeta_k)\bigr), \tag{3.18}$$

$C_{i_1,\cdots,i_n}$'s are complex numbers, and $n_k = \frac{m_k}{2}$ if m_k is even and $n_k = \frac{(m_k+1)}{2}$ if m_k is odd.

Next, we let $u = (u_1, \cdots, u_n)$ be a unit vector in $\mathbf{C}^n$. Then there exists a i $(1 \le i \le n)$ such that $u_i \ge \frac{1}{\sqrt{n}}$. For this fixed i, we have, by (3.17), that for $w \in \mathbf{C}$,

$$F_i(w) := f_i(\zeta_k + \sqrt{n}uw) = (\sqrt{n})^{m_k} c_k u_j^{m_k} w^{m_k} + \eta_k w^{s_k} + \sum_{j > s_k} b_j w^j, \quad (3.19)$$

where $s_k \ge \frac{3m_k}{2}$ is an integer, η_k and b_j are complex numbers.

Let $G_i(w) = \frac{F_i(w)}{w^{m_k}}$. Then $G_i(0) = (\sqrt{n})^{m_k} c_k u_j^{m_k} \ge c_k$. By (3.16) and (3.13) we have that for $|w| = 1$, $|G_i(w)| \le c(1+|z_k|)e^{2Aq_1(\zeta_k)}$, which is also true in $|w| \le 1$ by the maximum modulus theorem. Also let $H_i(w) = G_i(w) - G_i(0)$. Then by (3.19), we see that $H_i(w)$ has a zero at $w = 0$ of order at least $\frac{m_k}{2}$. Note that $|H_i(w)| \le 2c(1+|z_k|)e^{2Aq_1(\zeta_k)}$ on $|w| \le 1$. We have, by the Schwartz lemma, that

$$|H_i(w)| \le 2c(1 + |z_k|)e^{2Aq_1(\zeta_k)}|w|^{\frac{m_k}{2}}$$

on $|w| \le 1$. Thus, if $a \ne 0$ is a zero of $F_i(w)$ in $|w| \le 1$, then $G_i(a) = 0$ and thus that

$$2c(1 + |z_k|)e^{2Aq_1(\zeta_k)}|a|^{\frac{m_k}{2}} \ge |H_i(a)| = |G_i(0)| = (\sqrt{n})^{m_k} c_k u_i^{m_k} \ge c_k,$$

or,

$$|a|^{\frac{m_k}{2}} \ge (2c)^{-1}(1 + |\zeta_k|)^{-N-2} e^{-2(A+n)q_1(\zeta_k)} \ge \epsilon(1 + |\zeta_k|)^{-c} e^{-cq_1(\zeta_k)},$$

for some $\epsilon, c > 0$, in view of (3.18). If we let

$$d_u = \min\{1, \text{dist } (0, F_i^{-1}(0) \setminus \{0\})\}.$$

Then we have that

$$d_u^{m_k} \geq \epsilon(1+|\zeta_k|)^{-c}e^{-cq_1(\zeta_k)} := (2\sqrt{n}d_k)^{m_k}. \tag{3.20}$$

Note that $G_i(w)$ has no zero in $|w| \leq 2d_k \leq d_u$ by the construction of d_u. Recall the following result from the Carathéodory theorem (see e.g. [Le]): If h is holomorphic and has no zero in $|w| \leq R$ with $h(0) = 1$, then $\log|h(w)| \geq -\frac{2r}{R-r}\log\max_{|w|=R}\{|h(w)|\}$ for $|w| \leq r < R$. Applying it to $G_i(w)$ in $|w| \leq 2d_k$ we deduce that for $|w| \leq d_k$

$$\log|\frac{G_i(w)}{G_i(0)}| \geq -2\log(\max_{|w|=d_k}\{|\frac{G_i(w)}{G_i(0)}|\}),$$

which implies that

$$\log|G_i(w)| \geq -2\log(\max_{|w|=d_k}\{|G_i(w)\}) + 3\log|G_i(0)|$$

and so that

$$\begin{aligned}|G_i(w)| &\geq (\max_{|w|=d_k}\{|G_i(w)\})^{-2}|G_i(0)|^3\\ &\geq c_k^3c^{-1}(1+|z_k|)^{-2}e^{-4Aq_1(\zeta_k)} \geq \epsilon(1+|z_k|)^{-c}e^{-cq_1(\zeta_k)},\end{aligned}$$

for some $\epsilon, c > 0$. By (3.20) we have, for $|w| = d_k$, that

$$\begin{aligned}|F_i(w)| &= |w^{m_k}G_i(w)| = d_k^{m_k}|G_i(w)|\\ &\geq (\frac{1}{2\sqrt{n}})^{m_k}\epsilon(1+|z_k|)^{-c}e^{-cq_1(\zeta_k)}.\end{aligned}$$

On the other hand, by (3.17), (3.16), and by the Cauchy estimates,

$$\begin{aligned}c_k &= |\frac{1}{(m_k)!}\frac{\partial^{m_k}f_i(\zeta_k)}{\partial z_i^{m_k}}|\\ &\leq \frac{c}{(2\sqrt{n})^{m_k}}\max_{P_k}\{|f_i(z)|\} \leq \frac{c}{(2\sqrt{n})^{m_k}}(1+|z_k)e^{2Aq(z_k)},\end{aligned}$$

where $P_k = z \in \mathbf{C}^n : |z_j - \zeta_{k,j}| < 2\sqrt{n}, 1 \le j \le n$. Hence,

$$(\frac{1}{2\sqrt{n}})^{m_k} \ge \epsilon(1 + |z_k|)^{-c} e^{-cq_1(z_k)}$$

and we finally obtain that on $|w| = d_k$,

$$|f_i(\zeta_k + \sqrt{n}uw)| = |F_i(w)| \ge \epsilon e^{-cq(\zeta_k)},$$

for some $\epsilon, c > 0$ independent of u and k, where $q(z) = \log(1 + |z|) + q_1(z) = o\{p(z)\}$ by (2.1) and the fact that $q_1(z) = o\{p(z)\}$.

Since the above u is an arbitrary unit vector, we thus have showed that

$$|f(\zeta_k + z)| \ge |f_i(\zeta_k + z)| \ge \epsilon e^{-cq(\zeta_k)}$$

for $|z| = \sqrt{n}d_k$. By virtue of (3.13) we have that

$$|f(z)| \ge \epsilon e^{-Cq(z)}$$

for some $\epsilon, C > 0$ on $|z - \zeta_j| = \sqrt{n}d_k$. Note that $\sqrt{n}d_k \le \frac{1}{2}d_u \le \frac{1}{2}$ in view of (3.20). We have thus showed that the connected component U_k of $S(f; \epsilon, c) = \{z \in \mathbf{C}^n : |f(z)| < \epsilon e^{-Cq(\zeta_k)}\}$ containing ζ_k must be completely contained in the ball $|z - \zeta_k| = \sqrt{n}d_k$, which has diameter at most 1 and does not contain any other points of V. This shows the necessity of the theorem when $n = N$. If $N > n$, we can easily add $N - n$ entire functions $f_{n+1}, \cdots, f_N \in A_p^0(\mathbf{C}^n)$ satisfying $V \subset f_j^{-1}(0), n + 1 \le j \le N$. Let $F = (f_1, f_2, \cdots, f_N)$. Then $S_q(F; \epsilon, C) \subseteq S_q(f, \epsilon, C)$. Thus, the mapping F satisfies the conclusion of the theorem.

The proof of the sufficiency is much simpler. Suppose that $\{a_{k,I}\}_{k \in \mathbf{N}, 0 \le |I| < m_k} \in A_p^0(V)$ be any given sequence. It suffices to find an entire function $F \in A_p^0(\mathbf{C}^n)$ such that $\frac{\partial^{|I|} F(z_k)}{I! \partial z^I} = a_{k,I}$ for all $k \in \mathbf{N}, 0 \le |I| < m_k$.

For any integer m there exists a $c_m \geq 1$ such that $\sum_{|I|=0}^{m_k-1} |a_{k,I}| < c_m e^{\frac{1}{m}p(z_k)}$ for each $k \in \mathbf{N}$ since $\{a_{k,I}\} \in A_p^0(V)$ and, meanwhile, $|f(z)| < c_m e^{\frac{1}{m}p(z_k)}$ for each $z \in \mathbf{C}^n$, since $f \in A_p^0(\mathbf{C}^n)$. Thus,

$$\sum_{|I|=0}^{m_k-1} |a_{k,I}| < \exp\bigl(\inf_m \{\log c_m + \frac{1}{m}p(z_k)\}\bigr) \tag{3.21}$$

for each k and

$$|f(z)| < \exp\bigl(\inf_m \{\log c_m + \frac{1}{m}p(z)\}\bigr) \tag{3.22}$$

for $z \in \mathbf{C}^n$. Define

$$q_a(z) = \inf_m \{\log c_m + \frac{1}{m}p(z)\}, \quad \alpha(z) = \max\{q(z), g_a(z)\}.$$

Then $\alpha(z) = o\{p(z)\}$. We recall the following theorem in [BMT, 1.7 and 1.8]: For any continuous and increasing function $\omega(r)$, if $\omega(r)$ satisfies (2.1) and (2.2), and $\omega(e^r)$ is convex, then for any function $h(r) : [0, \infty) \to [0, \infty)$ satisfying that $h(r) = o(\omega(r))$ there exists an increasing function $g(r)$ such that g satisfies (2.1) and (2.2), $g(e^r)$ is convex, and $h(r) = o\{g(r)\}$ and $g(r) = o\{\omega(r)\}$. Applying this result with $\omega = p$ and $h = \alpha$ we obtain an increasing function $q_a(r)$ satisfying (2.1), (2.2), $q_a(e^r)$ is convex, and $\alpha(r) = o\{q_a(r)\}$ and $q_a(r) = o\{p(r)\}$. Note the fact that $H \circ u$ is plurisubharmonic if H is convex and increasing, and u is plurisubharmonic. We thus have that $q_a(|z|) = q_a(e^{\ln |z|})$ is plurisubharmonic and so that $q_a(|z|)$ is a weight. It is clear, by (3.21) and (3.22), that $\sum_{|I|=0}^{m_k-1} |a_{k,I}| < e^{q_a(z_k)}$ for each k, and $|f(z)| < e^{q_a(z)}$ for each $z \in \mathbf{C}^n$, which implies that $f \in A_{q_a}(\mathbf{C}^n)$. Also, it is obvious that $S_{q_a}(f, \epsilon, C) \subseteq S_q(f, \epsilon, C)$. Thus, by the hypotheses of the theorem, each connect component of $S_{q_a}(f, \epsilon, C)$ contains at most one point in V and such a component has diameter at most 1. Let U_k be the connected component of $S_{q_a}(f, \epsilon, C)$ containing z_k. We define an analytic function $\lambda : S_{q_a}(f; \epsilon, C) \to \mathbf{C}$ by

$$\lambda(z) = \begin{cases} \sum_{|I|=0}^{m_k-1} a_{k,I}(z - z_k)^I, & \text{if } z \in U_k; \\ 0, & \text{if } z \in S_{q_a}(f, \epsilon, C) \setminus \cup_{k \in \mathbf{N}} U_k. \end{cases}$$

Then it is clear that $\frac{1}{I!}\frac{\partial^{|I|}\lambda(z_k)}{\partial z^I} = a_{k,I}$ for all $k \in \mathbf{N}$ and all $0 \le |I| < m_k$. Moreover, on U_k we have that $|z - z_k| \le 1$, since the diameter of U_k is at most 1, and thus that

$$|\lambda(z)| = \sum_{|I|=0}^{m_k-1} |a_{k,I}| \le e^{q_a(z_k)} \le e^{Aq_a(z)+B} \tag{3.23}$$

for some $A, B > 0$ by virtue of the property (2.2) of a weight, which implies that $q_a(w) \le Aq_a(z) + B$ whenever $|w - z| \le 1$. By the definition of λ, the estimate (3.23) holds for all z in $S_{q_a}(f; \epsilon, C)$. We then use the following result in [BT, Theorem 2.2]: If λ is analytic and satisfies that $|\lambda(z)| \le e^{Aq_a(z)+B}$ for some $A, B > 0$ on $S_q(f, \epsilon, C)$, where q is a weight and $f = (f_1, f_2, \cdots, f_m) : \mathbf{C}^n \to \mathbf{C}^m$ is an entire holomorphic mapping with $f_j \in A_q(\mathbf{C}^n)$, then there exist an entire function $F \in A_q(\mathbf{C}^n)$ such that $F(z) = \lambda(z)$ on the variety $f(z) = 0$. Applying this result to our function λ, we obtain a function $F \in A_{q_a}(\mathbf{C}^n) \subset A_p^0(\mathbf{C}^n)$ such that $F(z) = \lambda(z)$ on $f^{-1}(0) \supseteq V$. In particular, $\frac{1}{I!}\frac{\partial^{|I|}F(z_k)}{\partial z^I} = \frac{1}{I!}\frac{\partial^{|I|}\lambda(z_k)}{\partial z^I} = a_{k,I}$ for all $k \in \mathbf{N}$ and all $0 \le |I| < m_k$. This shows that V is an interpolating variety for $A_p^0(\mathbf{C}^n)$, and thus concludes the proof.

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