INTERPOLATING MULTIPLICITY VARIETIES IN $\mathbf{C}^n$

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Abstract: We shall give a necessary and sufficient condition for a discrete multiplicity variety in $\mathbf{C}^n$ to be an interpolating variety for weighted spaces of entire functions.

§1. **Introduction**. In this paper, we shall consider when a discrete multiplicity variety in $\mathbf{C}^n$ is an interpolating variety for entire functions with growth conditions.

Let f be an entire function and $\{\zeta_k\}$ a discrete set in $\mathbf{C}^n$. Then we have the following Taylor expansion at each ζ_k:

$$f(z) = \sum_{|I|=0}^{\infty} f_{k,I}(z-\zeta_k)^I, \qquad z = (z_1, z_2, \cdots, z_n) \in \mathbf{C}^n,$$

where (and throughout the paper) $f_{k,I} := \frac{1}{I!}\frac{\partial^{|I|} f(\zeta_k)}{\partial z^I}$, $I := (i_1, \cdots, i_n) \in (\mathbf{Z}^+)^n$ is a multi-index, $\mathbf{Z}^+ = \{0, 1, 2 \cdots\}$, and $|I| = i_1 + i_2 + \cdots + i_n$.

Let p be a plurisubhamonic weight function in $\mathbf{C}^n$ and $\{m_k\}$ a sequence of positive integers. We consider the following interpolation problem with multiplicities: under what conditions is it true that for any multi-indexed sequence $\{a_{k,I}\}_{k\in \mathbf{N}, 0\le |I|<m_k}$ of complex numbers satisfying that

$$\sum_{|I|=0}^{m_k-1} |a_{k,I}| < Ae^{Bp(\zeta_k)}, \qquad k \in \mathbf{N} := \{1, 2, \cdots\},$$

for some constants $A, B > 0$, there exists an entire function in $\mathbf{C}^n$ such that

$$f_{k,I} = a_{k,I}, \quad \text{for} \quad k \in \mathbf{N}, 0 \le |I| < m_k, \tag{1.1}$$

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where $I := (i_1, \cdots, i_n) \in (\mathbf{Z}^+)^n$ is a multi-index, and f satisfies the same kind of growth condition, namely, $f \in A_p(\mathbf{C}^n)$, or equivalently,

$$|f(z)| < A' e^{B' p(z)}, \qquad z \in \mathbf{C}^n$$

for some constants $A', B' > 0$. We will then say that $V := \{(\zeta_k, m_k)\}$ is an interpolating (multiplicity) variety for the weight p, or for $A_p(\mathbf{C}^n)$. Note that the condition (1.1) means that f has a prescribed finite collection of Taylor coefficients at each ζ_k. In the special case that $m_k = 1$ for all k, (1.1) simply means that f takes prescribed values at each ζ_k.

The problem has been studied by many people due to its applications to other subjects such as harmonic analysis, number theory and systems research (see [BG], [BKS], [BL1], [BL2], [BT1], [BT2], [BT3], [EM], [L], [S], etc.). In particular, some interesting interpolation results were obtained by Berenstein and Taylor in [BT1] when $V = \{\zeta_k\}$ is a complete intersection defined by so-called slowly decreasing entire functions, where a vector function $F = (F_1, \cdots, F_n)$, $F_j \in A_p(\mathbf{C}^n)$, is called slowly decreasing if and only if there exist $\epsilon, C, C_1, C_2 > 0$ such that

(i) the connected components of the set $S(F; \epsilon, C)$ are bounded, where

$$S(F; \epsilon, C) = \{z \in \mathbf{C}^n : |F(z)| := (\sum_{j=1}^{n} |F_j(z)|^2)^{\frac{1}{2}} < \epsilon \exp(-Cp(z))\}; \qquad (1.2)$$

(ii) if Ω is a component of $S(F; \epsilon, C)$, then $p(z) \le C_1 p(w) + C_2$, for all $z, w \in \Omega$.

Note that the "tube" $S(F; \epsilon, C)$ plays an important role in interpolation. We refer the reader to [BT1] for a class of examples of slowly decreasing functions. With the above notion, the following interpolation result for a discrete set $\{\zeta_k\}$ without multiplicities was given in [BT1] :

Theorem A. *Let* $F = (F_1, \cdots, F_n)$, $F_j \in A_p(\mathbf{C}^n)$, *be slowly decreasing. Assume that the discrete set* $V = \{\zeta_k\}$ *is the zero set of* F *and each zero* ζ_k *is simple; that is,*

$$\det J_F(\zeta_k) \neq 0 \quad (J_F = \textit{Jacobian matrix of } F).$$

Then $V = \{\zeta_k\}$ *is an interpolating variety for* $A_p(\mathbf{C}^n)$ *if and only if there exist constants* $\epsilon, C > 0$ *such that each component of the "tube"* $S(F; \epsilon, C)$ *contains at most one point in* V.

An equivalent statement to this result in terms of the Jacobian of the defining functions was also given in [BT1]. Notice that Theorem A can only apply to the discrete set $V = \{\zeta_k\}$ (without multiplicity) in $\mathbf{C}^n$ that is exactly the complete intersection of some slowly decreasing functions. It was asked in [BT1, p.213] whether or not it holds in general. Since in practice the multiplicity problem naturally arises, and the set V is generally not a complete intersection of some slowly decreasing functions, it is a natural problem to find necessary and sufficient interpolation conditions that can apply to any given multiplicity varieties in $\mathbf{C}^n$. When $n = 1$, various results in this direction have been known (see [BG], [BL2], [BT2], [S], etc.). When $n > 1$, the problem has been recently considered in [BL1] for discrete varieties $\{\zeta_k\} \subset \mathbf{C}^n$ but without multiplicities, where a necessary and sufficient interpolation condition in terms of the "directional derivatives" of defining functions was found. In this paper, we shall give an answer to the above problem for general multiplicity varieties. A necessary and sufficient interpolation condition, which applies to general multiplicity varieties, will be given. It turns out that a multiplicity variety $V = \{(\zeta_k, m_k)\}$ is an interpolating variety for $A_p(\mathbf{C}^n)$ if and only if there exist constants $\epsilon, C > 0$, and $m (\geq n)$ functions $f_1, \cdots, f_m$ such that these functions vanish at each ζ_k

with multiplicity $\geq m_k$, and each component of the "tube" $S(F;\epsilon,C)$ defined as in (1.2), where $F=(f_1,\cdots,f_m)$, contains at most one point ζ_k and such a component has diameter at most one (see Theorem 2.6).

We note that the main result in the paper is the necessary interpolation condition. If $V=\{(\zeta_k,m_k)\}$ is an interpolating variety for $A_p(\mathbf{C}^n)$, then the defining functions of V similar to those in [BL1] can be found, which however can not satisfy our requirements, since there is no proper minimum modulus theorem for holomorphic functions in several complex variables and thus the multiplicity m_k, which may be unbounded and very large (as large as $p(\zeta_k)$), makes it difficult to bound the defining vector function from below away from each point of the variety, which is however essential for our conclusion. The key ingredient for the proof is the construction of the defining functions $F_1,\cdots,F_n$ whose Taylor expansions at each point of the variety have "enough gaps" (see the proof of Theorem 2.6). These gaps will enable us to show that the vector function $F=(F_1,\cdots,F_n)$ is "fairly" large away from each point ζ_k, which is crucial in the proof.

§2. Definitions and Results.

The following definitions and notations will be used throughout the paper.

Definition 2.1. A plurisubharmonic function $p:\mathbf{C}^n\to[0,\infty)$ is called a weight (function) if it satisfies the following conditions:

$$\log(1+|z|^2)=O\{p(z)\} \tag{2.1}$$

and there exist positive constants D_1 and D_2 such that $|z-w|\leq 1$ implies that

$$p(z)\leq D_1p(w)+D_2. \tag{2.2}$$

Definition 2.2. Let $A(\mathbf{C}^n)$ be the ring of all entire functions on $\mathbf{C}^n$.Then

$$A_p(\mathbf{C}^n) = \{f \in A(\mathbf{C}^n) : |f(z)| \leq A\exp(Bp(z)) \text{ for some } \quad A, B > 0\}.$$

We note that it is not the specific conditions on p which are important, but rather their consequences for the ring $A_p(\mathbf{C}^n)$. The (2.1) implies that $A_p(\mathbf{C}^n)$ contains the polynomials, and (2.2) implies that $A_p(\mathbf{C}^n)$ is closed under differentiation (see Lemma 3.3). One can replace the condition (2.2) by the following Hörmander's condition [H] : there exist four positive constants $c_1, \cdots, c_4$ such that $|z - w| \leq e^{-c_1 p(w) - c_2}$ implies that $p(z) \leq c_3 p(w) + c_4$. We use (2.2) only for the sake of convenience.

Example 2.3. The two basic examples of such weight functions are $p(z) = |z|^\rho (\rho > 0)$ and $p(z) = |\Im z| + \log(1 + |z|^2)$ corresponding to the space $A_p(\mathbf{C}^n)$ of all entire functions of order$\leq \rho$ and finite type and the space $\hat{\mathcal{E}}'(\mathbf{R}^n)$ of Fourier transforms of distributions with compact support in $\mathbf{R}^n$.

Let $f \not\equiv 0$ be a holomorphic function on an open connected neighborhood of $\zeta \in \mathbf{C}^n$. Then a series $f(z) = \sum_{j=\nu}^{\infty} \mathcal{P}_j(z - \zeta)$ converges uniformly on some neighborhood of ζ and represents f on this neighborhood. Here $\mathcal{P}_j$ is a homogeneous polynomial of degree j and $\mathcal{P}_\nu \not\equiv 0$. The nonnegative integer ν, uniquely determined by f and ζ, is called the zero multiplicity, or zero divisor of f at ζ, denoted by $\mathrm{div}_f(\zeta)$.

Let $V = \{(\zeta_k, m_k)\}$ be a multiplicity variety; that is, a discrete set $\{\zeta_k\} \subset \mathbf{C}^n$ with $|\zeta_k| \to \infty$ together with a sequence $\{m_k\}$ of positive integers. Associated to V, there is a unique closed ideal in $A(\mathbf{C}^n)$,

$$J = J(V) := \{f \in A(\mathbf{C}^n) : \mathrm{div}_f(\zeta_k) \geq m_k, \forall k\}.$$

Two entire functions g, h in $\mathbf{C}^n$ can be identified modulo J if and only

$$\frac{\partial^{|I|} g(\zeta_k)}{\partial z^I} = \frac{\partial^{|I|} h(\zeta_k)}{\partial z^I}, 0 \leq |I| < m_k, k \in \mathbf{N},$$

here and throughout the paper, we use I to denote a muti-index; that is, $I = (i_1, \cdots, i_n) \in (\mathbf{Z}^+)^n$. The quotient space $A(\mathbf{C}^n)/J$ can be identified to the space $A(V)$ of all sequences $\{a_{k,I}\}_{k\in\mathbf{N}, 0\le|I|<m_k}$ of complex numbers, which can be described as "analytic functions " on V. The map

$$\rho : \rho(f) = \{\frac{\partial^{|I|} f(\zeta_k)}{\partial z^I}\}_{k\in\mathbf{N}, 0\le|I|<m_k}$$

is the natural restriction map from $A(\mathbf{C}^n)$ into $A(V)$.

Definition 2.4. Let $V = \{(\zeta_k, m_k)\}$ be a multiplicity variety on $\mathbf{C}^n$. Then we define

$$A_p(V) = \{a = \{a_{k,I}\} \in A(V) : \exists A, B > 0, \sum_{|I|=0}^{m_k - 1} |a_{k,I}| \le A \exp(Bp(\zeta_k)), k \in \mathbf{N}\}.$$

It is easy to see that $\rho(A_p(\mathbf{C}^n) \subset A_p(V)$ (c.f. Lemma 3.3), but in general, the space $A_p(V)$ is too large. The interpolation problem with multiplicity stated in the introduction is simply to determine when ρ is surjective from $A_p(\mathbf{C}^n)$ to $A_p(V)$. That is, under what conditions, is it true that for any multi-indexed sequence $\{a_{k,I}\} \in A_p(\mathbf{C}^n)$ there exists an entire function $f \in A_p(\mathbf{C}^n)$ such that $f_{k,I} = a_{k,I}$ for any $k \in \mathbf{N}$ and $0 \le |I| < m_k$; i.e., f has a described finite collection of Taylor coefficients. When $m_k = 1$ for all k, then $f_{k,I} = a_{k,I}$ simply means that $f(\zeta_k) = a_k$, where $\{a_k\}$ is a sequence satisfying that $|a_k| \le Ae^{Bp(\zeta_k)}$ for some constants $A, B > 0$. When $n = 1$, then $f_{k,I} = a_{k,I}$ becomes that $\frac{f^{(l)}(\zeta_k)}{l!} = a_{k,l}$ for $k \in \mathbf{N}$ and $0 \le l \le m_k - 1$, where $\{a_{k,l}\}$ is a sequence satisfying that $\sum_{l=0}^{m_k-1} |a_{k,l}| \le Ae^{Bp(\zeta_k)}$ for some constants $A, B > 0$.

Definition 2.5. A multiplicity variety $V = \{(\zeta_k, m_k)\}$ is an interpolating variety for $A_p(\mathbf{C}^n)$ if the restriction map ρ is surjective from $A_p(\mathbf{C}^n)$ to $A_p(V)$.

Let $V = \{(\zeta_k, m_k)\}$ be a multiplicity variety. We use $V \subset F^{-1}(0)$, where $F = (F_1, F_2, \cdots, F_m)$, to denote that each F_j vanishes at ζ_k with multiplicity at least m_k; i.e., $\mathrm{div}_f(\zeta_k) \geq m_k$,$\forall k$. Sometimes, by abuse of language, we also refer to "a point " in a multiplicity variety $\{(\zeta_k, m_k)\}$ to mean the first entry ζ_k. Given $\epsilon, C > 0$, we define $S(F; \epsilon, C)$ by (1.2), which can be thought as a "tube" of the variety V.

We shall prove the following theorem:

Theorem 2.6. *Let $V = \{(\zeta_k, m_k)\}$ be a multiplicity variety in $\mathbf{C}^n$ and $m \geq n$ a positive integer. Then V is an interpolating variety for $A_p(\mathbf{C}^n)$ if and only if there exist m functions $f_1, f_2, \cdots, f_m$ in $A_p(\mathbf{C}^n)$ and two constants $\epsilon, C > 0$ such that $V \subset F^{-1}(0)$, where $F = (f_1, f_2, \cdots, f_m)$, and each connected component of $S(F; \epsilon, C) := \{z \in \mathbf{C}^n : |F(z)| < \epsilon e^{-Cp(z)}\}$ contains at most one point in V and such a component has diameter at most one.*

As an application of the above theorem, we have the following corollaries, the proof of which need both the necessary and sufficient conditions in Theorem 2.6. Corollary 2.7 gives an affirmative answer to a question in [BT3], when the variety under consideration is a multiplicity variety. The case when $m_k = 1$ for all k in the following corollaries appeared in [BL1].

Corollary 2.7. *Let $V = \{(\zeta_k, m_k)\}$ be an interpolating variety for $A_p(\mathbf{C}^n)$. Then V is also an interpolating variety for $A_q(\mathbf{C}^n)$ for any weight $q \geq p$.*

Corollary 2.8. *Let $p_j : \mathbf{C} \to [0, \infty)$ be weights in $\mathbf{C}$ and $V_j = \{(\zeta_{j,k}, m_{j,k})\}_{k=1}^{\infty}$ be an interpolating variety for p_j $(1 \leq j \leq n)$. Let $m_k = \min_{1 \leq j \leq n}\{m_{j,k}\}$, $\zeta_k = (\zeta_{1,k,}, \zeta_{2,k}, \cdots, \zeta_{n,k})$, and $p(z) = p_1(z_1) + \cdots + p_n(z_n)$ for $z = (z_1, z_2, \cdots, z_n)$. Then $V = \{(\zeta_k, m_k)\}$ is an interpolating variety for $A_p(\mathbf{C}^n)$.*

§3. Proofs of the Results.

In the following, we shall use A, B, C, ϵ to denote positive constants, the actual values of which may vary from one occurrence to the next.

To prove the results, we need the following lemmas.

Lemma 3.1. *Let* $V = \{(\zeta_k, m_k)\}$ *be an interpolating variety for* $A_p(\mathbf{C}^n)$. *Then given* $M > 0$ *there exist two constants* $l > 0$ *and* $\epsilon > 0$ *such that*

$$A_{p,l}(V) \supset \{a = \{a_{k,I}\}_{k\in\mathbf{N},|I|<m_k} : ||a|| < \epsilon\},$$

where $||a|| = \sup_{k\in\mathbf{N}}\{\sum_{|I|=0}^{m_k-1} |a_{k,I}|e^{-Mp(\zeta_k)}\}$ *and* $A_{p,l}(V) =$

$$= \{a_f := \{f_{k,I}\}_{k\in\mathbf{N},|I|<m_k} : f \in A(\mathbf{C}^n), |f(z)| \leq le^{lp(z)}, z \in \mathbf{C}^n \, and \, ||a_f|| \leq 1\}.$$

Proof. Let $\mathcal{A} = \{a = \{a_{k,I}\}_{k\in\mathbf{N},|I|<m_k} : ||a|| \leq 1\}$. Then it is easy to check that $\mathcal{A}$ is complete under the metric induced by the norm $||a||$. Let

$$A_{p,l}(\mathbf{C}^n) = \{f \in A_p(\mathbf{C}^n) : |f(z)| \leq le^{lp(z)}, z \in \mathbf{C}^n\}.$$

Then we have that

$$A_{p,l}(V) = \{a_f := \{f_{k,I}\}_{k\in\mathbf{N},|I|<m_k} : f \in A_{p,l}(\mathbf{C}^n) \quad \text{and} \quad ||a_f|| \leq 1\}.$$

Because V is an interpolating variety for $A_p(\mathbf{C}^n)$, for any sequence $a = \{a_{k,I}\} \in \mathcal{A}$, there exists a $f \in A_{p,l}(\mathbf{C}^n)$ for some l such that $f_{k,I} = a_{k,I}$ for $k \in \mathbf{N}$ and $|I| < m_k$. That is, $a \in A_{p,l}(V)$. This shows that $\mathcal{A} = \cup_{l=1}^{\infty} A_{p,l}(V)$.

One can check that each $A_{p,l}(V)$ is a closed subset of $\mathcal{A}$. In fact, if f_j is a sequence in $A_{p,l}(\mathbf{C}^n)$ such that $(f_j)_{k,I} \to a \in \mathcal{A}$ as $j \to \infty$, then by the property of the weight p and using Montel's theorem (see e.g. [G]) we know that $\{f_j\}$ is a normal family. By passing to a subsequence, we can assume that $f_j \to f$ normally, where f is the limit function. By the Weierstrass theorem, $f \in A_{p,l}(\mathbf{C}^n)$ and moreover $\{f_{k,I}\} = a$. It follows that $a \in A_{p,l}(V)$ and thus that $A_{p,l}(V)$ is closed. Now by the Baire-category theorem we know that for some l, $A_{p,l}(V)$ has a non-empty interior. Therefore, there exists a $\epsilon > 0$ such that $A_{p,l}(V) \supset \{a = \{a_{k,I}\}_{k\in\mathbf{N},|I|<m_k} : ||a|| < \epsilon\}$. □

Lemma 3.2. *Let $V = \{(\zeta_k, m_k)\}$ be an interpolating variety for $A_p(\mathbf{C}^n)$. Then there exist two constants $l > 0$ and $\epsilon > 0$ and a sequence $\{f_k\}$ of entire functions such that*

$$|f_k(z)| \leq le^{lp(z)}, \quad z \in \mathbf{C}^n \quad \text{and} \quad k \in \mathbf{N} \tag{3.1}$$

and

$$(f_k)_{j,I} = 0, \forall j, |I| \leq m_j - 1, \text{except that} \frac{1}{(m_k-1)!}\frac{\partial^{m_k-1} f_k(\zeta_k)}{\partial z_1^{m_k-1}} = \epsilon. \tag{3.2}$$

Proof. It follows from Lemma 3.1. In fact, taking $M = 1$ in Lemma 3.1, we then obtain two constants $l > 0$ and $\epsilon > 0$ such that the space $A_{p,l}(V)$ contains the space $S := \{a = \{a_{k,I}\}_{k\in\mathbf{N},|I|<m_k} : ||a|| < \epsilon\}$ (see Lemma 3.1 for the notations). For each fixed k, consider the sequence $a_k = \{a_{k,I}\}_{k\in\mathbf{N},|I|<m_k}$ satisfying that $a_{j,I} = 0$ for all j and $0 \leq |I| < m_j$ except that $a_{k,I_k} = \epsilon$, where $I_k = (m_k - 1, 0, \cdots, 0) \in (\mathbf{Z}^+)^n$. Then it is clear that $a_k \in S$. Hence there exists entire functions f_k such that (3.1) holds and that $(f_k)_{j,I} = a_{j,I}$ for all j and $0 \leq |I| < m_j$. That is (3.2) holds. □

Lemma 3.3 [BT1]. *If $f \in A_p(\mathbf{C}^n)$, say $|f(z)| \leq Ae^{Bp(z)}$ for $z \in \mathbf{C}^n$, then there exist $A', B' > 0$ depending only on A, B and the weight p, but not on f, such that*

$$\sum_{|I|=0}^{\infty} \frac{1}{I!}\left|\frac{\partial^{|I|} f(z)}{\partial z^I}\right| \leq A'e^{B'p(z)}, z \in \mathbf{C}^n.$$

Lemma 3.4 (Schwarz, [G]). *If f is holomorphic in an open neighborhood of a closed ball $\bar{B}(\zeta, r)$ in $\mathbf{C}^n$ centered at ζ and with radius r, $|f(z)| \leq M$ for $z \in B(\zeta, r)$, and $\frac{\partial^{|I|} f}{\partial z^I}(\zeta) = 0$ whenever $|I| < m$ for some $m \in \mathbf{N}$, then $|f(z)| \leq M r^{-m} |z - \zeta|^m$ for $z \in \bar{B}(\zeta, r)$.*

If $V = \{(\zeta_k, m_k)\}$ is an interpolating variety for $A_p(\mathbf{C}^n)$, then V has the following three properties. In fact, we have the following result:

Lemma 3.5. *Let $V = \{(\zeta_k, m_k)\}$ be a multiplicity variety in $\mathbf{C}^n$ and $\{f_k\}$ a sequence of entire functions satisfying (3.1) and (3.2) for some $l, \epsilon > 0$. Then*

(i) $m_k \leq Ap(\zeta_k) + B$ for some $A, B > 0$;

(ii) $\eta_k := \inf_{j \neq k}\{|\zeta_k - \zeta_j|\} \geq \epsilon e^{-Cp(\zeta_k)}, \forall k$, for some constants $\epsilon, C > 0$;

and

(iii) $\sum_{k=1}^{\infty} e^{-Mp(\zeta_k)} < \infty$ for some $M > 0$.

Proof. For each fixed k, we have by the Cauchy Theorem,

$$\frac{1}{(m_k - 1)!} \frac{\partial^{m_k - 1} f_k(\zeta_k)}{\partial z_1^{m_k - 1}} = \left(\frac{1}{2\pi i}\right)^n \int \frac{f_k(z) dz_1 \wedge dz_2 \cdots \wedge dz_n}{(z_1 - \zeta_{1,k})^{m_k} (z_2 - \zeta_{2,k}) \cdots (z_n - \zeta_{n,k})},$$

where $\zeta_k = (\zeta_{1,k}, \zeta_{2,k}, \cdots, \zeta_{n,k})$, and the integral is taken over the distinguished boundary P_k of the polydisc $\{z = (z_1, \cdots, z_n) \in \mathbf{C}^n : |z_j - \zeta_{j,k}| \leq e, j = 1, 2, \cdots, n\}$. Therefore by (3.2)

$$\epsilon \leq \frac{A}{e^{m_k}} \max_{z \in P_k}\{|f(z)|\} \leq \frac{A}{e^{m_k}} e^{Bp(\zeta_k)}$$

and so that $e^{m_k} \leq A e^{Bp(\zeta_k)}$, or $m_k \leq A + Bp(\zeta_k)$ for some constants $A, B > 0$ independent of k. That is, (i) holds.

For each k, there exists a j such that $d_k = |\zeta_k - \zeta_j|$. We may assume that $m_j \leq m_k$ (otherwise, exchange the positions of k and j and do obvious modification in the following argument). Set $F_k(z) = \frac{1}{(m_j - 1)!} \frac{\partial^{m_j - 1} f_j(z)}{\partial z_1^{m_j - 1}}$. Then by (3.2) we know that $F_k(\zeta_k) = 0$, but $F_k(\zeta_j) = \frac{1}{(m_j - 1)!} \frac{\partial^{m_j - 1} f_j(\zeta_j)}{\partial z_1^{m_j - 1}} = \epsilon$. By

(3.1) and using Lemma 3.3, we know that $|F_k(z)| \leq Ae^{Bp(\zeta_k)}$ for $|z-\zeta_k| \leq 1$, where $A, B > 0$ are some constants independent of k. Next by Lemma 3.4, we have that $|F_k(z)| \leq Ae^{Bp(\zeta_k)}|z-\zeta_k|$ for $|z-\zeta_k| \leq 1$. In particular, if $|\zeta_j - \zeta_k| \leq 1$, we then have that $|F_k(\zeta_j)| \leq Ae^{Bp(\zeta_k)}|\zeta_j - \zeta_k|$ and thus that $\epsilon \leq Ae^{Bp(\zeta_k)}d_k$; that is, $d_k \geq \epsilon e^{-Cp(\zeta_k)}$ for some constants $\epsilon, C > 0$. This inequality is obviously true if $|\zeta_j - \zeta_k| > 1$. Thus, (ii) follows. The last conclusion follows from (ii) and the following result in [BL1]: Let $\{z_k\}$ be a discrete set in $\mathbf{C}^n$ with $\delta_k := \inf_{j\neq k}\{|z_k - z_j|\} \geq \epsilon e^{-p(\zeta_k)}, \forall k$, for some constants $\epsilon, C > 0$, then there exists a $M > 0$ such that $\sum_{k=1}^{\infty} e^{-Mp(\zeta_k)} < \infty$. This completes the proof. □

Proof of Theorem 2.6. We first prove the necessity. Let

$$A_{p,l}(\mathbf{C}^n) = \{f \in A_p(\mathbf{C}^n) : |f(z)| \leq le^{lp(z)}, z \in \mathbf{C}^n\}$$

and

$$A_{p,l}(V) = \{\{f_{k,I}\}_{k\in\mathbf{N},|I|<m_k} : f \in A_{p,l}(\mathbf{C}^n), \sup_{k\in\mathbf{N}}\{\sum_{|I|=0}^{m_k-1} |f_{k,I}|e^{-Mp(\zeta_k)}\} \leq 1\},$$

where $M > 0$ is a constant. By Lemma 3.1, there exists a positive integer l and a ϵ_0 such that

$$A_{p,l}(V) \supset \{\{a_{k,I}\}_{k\in\mathbf{N},|I|<m_k} : \sup_{k\in\mathbf{N}}\{\sum_{|I|=0}^{m_k-1} |a_{k,I}|e^{-Mp(\zeta_k)}\} < \epsilon_0\}.$$

Therefore, for each $1 \leq j \leq n$ we can obtain sequences of entire functions $\{g_{j,k}\}$ and $\{h_{j,k}\}$ with $g_{j,k}, h_{j,k} \in A_{p,l}(\mathbf{C}^n)$ for any $k \in \mathbf{N}$ and $1 \leq j \leq n$, such that

$$(g_{j,k})_{i,I} = 0, \quad \forall i, |I| < m_i - 1 \text{ except that } \frac{\partial^{[\frac{m_k}{2}]} g_{j,k}(\zeta_k)}{\partial z_j^{[\frac{m_k}{2}]}} = \epsilon_0 e^{Mp(\zeta_k)}, \quad (3.3)$$

and

$$(h_{j,k})_{i,I} = 0, \quad \forall i, |I| < m_i - 1 \text{ except that } \frac{\partial^{m_k-1-[\frac{m_k}{2}]} h_{j,k}(\zeta_k)}{\partial z_j^{m_k-1-[\frac{m_k}{2}]}} = \epsilon_0 e^{Mp(\zeta_k)}, \tag{3.4}$$

where $[x]$ denotes the biggest integer that is less than or equal to x. We define, for $1 \leq j \leq n$, the following functions

$$f_j(z) = \sum_{k=1}^{\infty} (z_j - \zeta_{j,k}) g_{j,k}(z) h_{j,k}(z) e^{-2Mp(\zeta_k)}, \tag{3.5}$$

where $z = (z_1, \cdots, z_n)$ and $\zeta_k = (\zeta_{1,k}, \cdots, \zeta_{n,k})$. It is clear that $\mathrm{div}_{f_j}(\zeta) \geq m_k$ and so that $V \subset F^{-1}(0)$, where $F = (f_1, \cdots, f_n)$. We claim that $f_j \in A_p(\mathbf{C}^n)$ for each $1 \leq j \leq n$. In fact, since $g_{j,k}, h_{j,k} \in A_{p,l}(\mathbf{C}^n)$ for any $k \in \mathbf{N}$, we have that for $z \in \mathbf{C}^n$,

$$|g_{j,k}(z)| \leq l e^{lp(z)}, \quad |h_{j,k}(z)| \leq l e^{lp(z)}.$$

Also, by the property (2.1) of the weight function p, we have that

$$|z_j - \zeta_{j,k}| \leq A e^{Bp(z)} + A e^{Bp(\zeta_k)}$$

for some constants $A, B > 0$. Therefore, we deduce that

$$(z_j - \zeta_{j,k}) g_{j,k}(z) h_{j,k}(z) e^{-2Mp(\zeta_k)} \leq A e^{Bp(z)} e^{(C-2M)p(\zeta_k)}$$

for some constants $A, B, C > 0$. By Lemma 3.2 and Lemma 3.5 (iii), taking a M sufficiently large, we see that the series (3.5) is uniformly convergent in compact sets of $\mathbf{C}^n$, and moreover $|f_j(z)| \leq A e^{Bp(z)}, z \in \mathbf{C}^n$ for some constants $A, B > 0$; that is , $f_j \in A_p(\mathbf{C}^n)$.

Next we show that there are positive constants ϵ, C such that the "tube" $S(F; \epsilon, C)$ satisfies the conclusion in the theorem. To this end, let $k > 0$ and let $u = (u_1, \cdots, u_n)$ be a unit vector in $\mathbf{C}^n$. Then there exists a j $(1 \leq j \leq n)$

such that $u_j \geq \frac{1}{\sqrt{n}}$. For this fixed j, consider the Taylor expansion of $f_j(z)$ at ζ_k. Noticing that

$$2m_k - [\frac{m_k}{2}] \leq [\frac{m_k}{2}] + m_k + 1 \leq 2m_k,$$

we can verify that, in view of (3.5), (3.3) and (3.4),

$$f_j(z) = \epsilon_0^2(z_j - \zeta_{j,k})^{m_k} + \sum_{|I| \geq 2m_k - [\frac{m_k}{2}]}^{\infty} C_I(z - \zeta_k)^I,$$

where C_I's are complex numbers. Note that there is a "gap" between the powers of the first and other terms in the above expansion, from which we have that, for $w \in \mathbf{C}$,

$$F_j(w) := f_j(\zeta_k + uw) = \epsilon_0^2 u_j^{m_k} w^{m_k} + \eta_k w^{2m_k - [\frac{m_k}{2}]} + \sum_{j > 2m_k - [\frac{m_k}{2}]} b_j w^j, \quad (3.6)$$

where η_k and b_j are complex numbers. By Lemma 3.5 (i), we deduce that

$$u_j^{m_k} \geq (\frac{1}{\sqrt{n}})^{m_k} \geq \epsilon e^{-Cp(\zeta_k)} \quad (3.7)$$

for some constants $\epsilon, C > 0$. Let $d_u = \min\{1, \text{dist } (0, F_j^{-1}(0) \setminus \{0\})\}$ and set $G_j(w) = \frac{F_j(w)}{w^{m_k}}$. Then $|G_j(w)| \leq Ae^{Bp(\zeta_k)}$ for some constants $A, B > 0$ on $|w| = 1$ and thus in $|w| \leq 1$ by the maximum modulus theorem. Also let $H_j(w) = G_j(w) - G_j(0)$. Then by (3.6), we see that $H_j(w)$ has a zero at $w = 0$ of order at least $m_k - [\frac{m_k}{2}]$. Note that $|H_j(w)| \leq Ae^{Bp(\zeta)}$ for some constants $A, B > 0$ on $|w| \leq 1$. We have , by Lemma 3.4, that

$$|H_j(w)| \leq Ae^{B(\zeta_k)}|w|^{m_k - [\frac{m_k}{2}]}$$

on $|w| \leq 1$. Thus, if $a \neq 0$ is a zero of $F_j(w)$ in $|w| \leq 1$, then $G_j(a) = 0$ and thus that $H_j(a) = G_j(0) = \epsilon_0^2 u_j^{m_k} \geq \epsilon_0^2 \epsilon e^{-Cp(\zeta_k)}$ by (3.6) and (3.7), from which it follows that

$$|a|^{m_k - [\frac{m_k}{2}]} \geq |G_j(0)|A^{-1}e^{-Bp(\zeta_k)} \geq \epsilon e^{-Cp(\zeta_k)}$$

for some constants $\epsilon, C > 0$. We thus have that $d_u^{m_k - [\frac{m_k}{2}]} \geq \epsilon e^{-Cp(\zeta_k)}$ for some constants $\epsilon, C > 0$. Now by virtue of the fact that $m_k - [\frac{m_k}{2}] \geq \frac{m_k}{2}$, we see that

$$d_u^{\frac{m_k}{2}} \geq d_u^{m_k - [\frac{m_k}{2}]} \geq \epsilon e^{-Cp(\zeta_k)}$$

and so that $d_u^{m_k} \geq \epsilon e^{-Cp(\zeta_k)}$ for some constants $\epsilon, C > 0$, or

$$d_u \geq \epsilon^{\frac{1}{m_k}} e^{-\frac{C}{m_k} p(\zeta_k)} := 2d_k. \tag{3.8}$$

Note that $G_j(w)$ has no zero in $|w| \leq 2d_k$ by the construction of d_u. (It may be worth to mention here the difference between d_u and $\eta_k := \inf_{j \neq k}\{|\zeta_k - \zeta_j|\}$. While G_j has no zero in $|w| \leq d_u$, G_j might have many zeros in $|w| \leq \eta_k$. Thus, the role of d_u can not be played by η_k.) We can now apply the Carathéodory theorem (see e.g. [Lv]) to $G_j(w)$ in $|w| \leq 2d_k$ to deduce that for $|w| \leq d_k$

$$\log |\frac{G_j(w)}{G_j(0)}| \geq -\frac{2d_k}{2d_k - d_k} \log(\max_{|w|=d_k} \{|\frac{G_j(w)}{G_j(0)}|\}),$$

or $|G_j(w)| \geq \epsilon e^{-Cp(\zeta_k)}$ for some constants $\epsilon, C > 0$. In particular, for $|w| = d_k$,

$$\begin{aligned} |F_j(w)| &= |w^{m_k} G_j(w)| \\ &\geq (\frac{1}{2})^{m_k} \{\epsilon^{\frac{1}{m_k}} e^{-\frac{C}{m_k} p(\zeta_k)}\}^{m_k} \epsilon e^{-Cp(\zeta_k)} \geq \epsilon e^{-Cp(\zeta_k)}, \end{aligned}$$

in view of (3.8) and the fact that $m_k \leq Ap(\zeta_k) + B$ for some $A, B > 0$ by Lemma 3.5 (i).

So far we have proved that for a given unit vector $u \in \mathbf{C}^n$, there exists a j $(1 \leq j \leq n)$ such that $|f_j(\zeta_k + uw)| \geq \epsilon e^{-Cp(\zeta_k)}$ on $|w| = d_k$, where the constants ϵ, C are independent of u and k. Therefore, for $z \in \mathbf{C}^n$ with $|z - \zeta_k| = d_k$, we always have that $|F(z)| = (\sum_{j=1}^n |f_j(z)|^2)^{\frac{1}{2}} \geq \epsilon e^{-Cp(\zeta)}$. Now consider the neighborhood $U_k := \{z \in \mathbf{C}^n : |z - \zeta_k| \leq d_k\}$ of ζ_k. By the above result, we know that $|F(z)| \geq \epsilon e^{-Cp(\zeta_k)}$ on ∂U_k. Recall that $S(F; \epsilon, C) =$

$\{z \in \mathbf{C}^n : |F(z)| < \epsilon e^{-Cp(\zeta_k)}\}$. Thus the connected component V_k of $S(F;\epsilon,C)$ containing ζ_k is clearly contained in U_k. By the construction of d_k, we see that U_k, and thus V_k, has diameter less than 1 and does not contain other points of V. (If $m > n$, we can easily add $m-n$ entire functions $f_{n+1},\cdots,f_m$ so that $f_1, f_2, \cdots, f_m$ satisfy the conclusion of the theorem.) This completes the proof of the necessity.

To prove the sufficiency, let V_k be the connected component of $S(F;\epsilon,C)$ containing ζ_k. Suppose that $\{a_{k,I}\} \subset A_p(V)$ be a given multi-indexed sequence with $\sum_{|I|=0}^{m_k-1} |a_{k,I}| \leq Ae^{Bp(\zeta_k)}$ for some constants $A, B > 0$. We define an analytic function $\lambda : S(F;\epsilon,C) \to \mathbf{C}$ by

$$\lambda(z) = \begin{cases} \sum_{|I|=0}^{m_k-1} a_{k,I}(z-\zeta_k)^I, & \text{if } z \in V_k; \\ 0, & \text{if } z \in S(F;\epsilon,C) \setminus \cup_{k\in\mathbf{N}} V_k. \end{cases}$$

Then it is clear that $\lambda_{k,I} = a_{k,I}$ for $k \in \mathbf{N}$ and $0 \leq |I| \leq m_k - 1$. Let $A_p(S(F;\epsilon,C)$ be the space of analytic functions g on $S(F;\epsilon,C)$ satisfying that $|g(z)| \leq Ae^{Bp(z)}$ on $S(F;\epsilon,C)$ for some $A, B > 0$. Then $\lambda \in A_p(S(F;\epsilon,C))$, since $|z-\zeta_k| \leq 1$ on V_k by the assumption. Now the sufficiency follows from the following theorem ([BT1]): If $\lambda \in A_p(S(F;\epsilon,C))$, then there exist an entire function $f \in A_p(\mathbf{C}^n)$, and $\epsilon_1, C_1 > 0$ such that $f(z)-\lambda(z) = \sum_{j=1}^m f_j(z)g_j(z)$ on $S(F;\epsilon,C)$, where $g_j \in A_p(S(F;\epsilon_1,C_1)$. Applying this theorem to our function λ, we obtain such a function $f \in A_p(\mathbf{C}^n)$. Then by checking the Taylor expansion of $f-\lambda$, we easily see that $f_{k,I} = \lambda_{k,I} = a_{k,I}$ for $k \in \mathbf{N}$ and $0 \leq |I| \leq m_k - 1$. This shows that V is an interpolating variety for $A_p(\mathbf{C}^n)$. □

Proof of Corollary 2.7. Since V is an interpolating variety for $A_p(\mathbf{C}^n)$, by the necessary condition of Theorem 2.6, there exist m functions $f_1, f_2, \cdots, f_m$ in $A_p(\mathbf{C}^n)$ and thus in $A_q(\mathbf{C}^n)$, and two positive constants $\epsilon, C > 0$ such that $V \subset F^{-1}(0)$, where $F = (f_1, f_2, \cdots, f_m)$, and each connected component of

$S(F;\epsilon,C) := \{z \in \mathbf{C}^n : |F(z)| < \epsilon e^{-Cp(z)}\}$ contains at most one point in V and such a component has diameter at most one. Let U_k and V_k be the component of $S(F;\epsilon,C)$ and the component of $S_1(F;\epsilon,C) := \{z \in \mathbf{C}^n : |F(z)| < \epsilon e^{-Cq(z)}\}$ that contain ζ_k, respectively. Then it is obvious that $V_k \subset U_k$ since $q \geq p$. Hence V_k does not contain other points in V and its diameter is at most one. Now by the sufficient condition, we know that V is an interpolating variety of $A_q(\mathbf{C}^n)$. □

Proof of Corollary 2.8. Since $V_j = \{(\zeta_{j,k}, m_{j,k})\}_{k=1}^{\infty}$ is an interpolating variety for $A_{p_j}(\mathbf{C})$ $(1 \leq j \leq n)$, by the necessary condition of Theorem 2.6, there exist $\epsilon, C > 0$ and functions F_j in $A_p(\mathbf{C})$ such that $V_j \subset F_j^{-1}(0)$ and the component U_k^j of $S(F_j;\epsilon,C) := \{z \in \mathbf{C} : |F_j(z)| < \epsilon e^{-Cp(z)}\}$ contains at most one point in V_j and such a component has diameter at most one. By taking smaller ϵ and larger C we can shrink the "tubes" $S(F_j;\epsilon,C)$ so that the diameter of U_k^j is at most $\frac{1}{\sqrt{n}}$. Set $F(z) = (F_1(z_1), F_2(z_2), \cdots, F_n(z_n))$, where $z = (z_1, \cdots, z_n) \in \mathbf{C}^n$. Then $V \subset F^{-1}(0)$ and $F_j(z_j) \in A_p(\mathbf{C}^n)$. Let U_k be the component of $S(F;\epsilon,C) := \{z \in \mathbf{C}^n : (|F_1(z_1)|^2 + \cdots + |F_n(z_n)|^2)^{\frac{1}{2}} < \epsilon e^{-Cp(z)}\}$. Then it is clear that $z \in U_k$ implies that $z_j \in U_k^j$, from which it follows that U_k can not contain other points in V. Moreover, if $z \in U_k$ then $|z - \zeta_k| = (|z_1 - \zeta_{1,k}|^2 + \cdots + |z_n - \zeta_{n,k}|^2)^{\frac{1}{2}} \leq 1$ in view of the fact that the diameter of U_k^j is at most $\frac{1}{\sqrt{n}}$. This shows that the diameter of U_k is at most 1. Now by the sufficient condition of Theorem 2.6, we know that V is an interpolating variety for $A_p(\mathbf{C}^n)$. □

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