

A GENERALIZATION OF RIESZ'S UNIQUENESS THEOREM

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ABSTRACT. There have been, over the last 8 years, a number of far reaching of the famous original F. and M. Riesz's uniqueness theorem that states that if a bounded analytic function in the unit disc of the complex plane \mathbb{C} has the same radial limit in a set of positive Lebesgue measure on its boundary, then the function has to be constant. First Beurling [B], considering the case of non-constant meromorphic functions mapping the unit disc on a Riemann surface of finite spherical area, was able to prove that if such a function showed an appropriate behavior in the neighborhood of the limit value where the function maps a set on the boundary of the unit disc, then those sets have capacity zero. Here the capacity considered is the logarithmic linear capacity. The author of the present note in [V], was able to weakened beurling condition on the limit value. Later Jenkins in [J], showed that in the presence of such a local condition on the limiting value, the global behavior of Riemann surface is irrelevant and at the same time he gave an improved and sharper condition.

Those results where quite restrictive in a two folded way, namely, they were in dimension $n = 2$ and the regularity requirements on the treated functions were quite strong, analyticity and meromorphicity. Koskela in [K], was able to remove those two restrictions by proving a uniqueness result for functions in $ACL^p(\mathbb{B}^n)$ for values of p in the interval $(1, n]$ and satisfying a condition on the limit value very similar in nature to the one of Jenkins in dimension 2. In particular, Koskela's result recovers Jenkins in the case $p = n = 2$. He proves that a continuous function in the Sobolev space $W^{1,p}(\mathbb{B}^n)$ (here \mathbb{B}^n is the unit ball of \mathbb{R}^n and $1 < p \leq n$) vanishes identically provided

$$\int_{\|u(x)-a\|<\epsilon} \|\nabla u(x)\|^p dx = O(\epsilon^p (\log(\frac{1}{\epsilon}))^{p-1})$$

as $\epsilon \rightarrow 0$ and there is a set E on $\partial\mathbb{B}^n$ of positive p -capacity such that each $x \in E$ is a terminal point of some rectifiable curve along which the function u tends to a .

Koskela also shows in his paper that this result is sharp in the sense that $(\log(\frac{1}{\epsilon}))^{p-1}$ can not be replaced by $(\log(\frac{1}{\epsilon}))^{p-1+\delta}$ for any positive δ (even if u is assumed to be continuous in the closure of \mathbb{B}^n).

§1. Introduction.

Mizuta in [M] showed that under the same hypothesis on the function u , if

$$\int_{\|u(x)-a\|<\epsilon} \|\nabla u(x)\|^p dx = O(\epsilon^p \phi(\epsilon))$$

as $\epsilon \rightarrow 0$, where ϕ is a positive nonincreasing function on the interval $(0, \infty)$ satisfying the following conditions

$$(1) \quad A^{-1} \phi(r) \leq \phi(r^2) \leq A \phi(r)$$

for every $r > 0$ and A a positive constant and

$$(2) \quad \int_0^1 [\phi(r)]^{\frac{1}{1-p}} r^{-1} dr = \infty,$$

then if there is a set E on $\partial\mathbb{B}^n$ of positive p -capacity such that each $x \in E$ is a terminal point of some rectifiable curve along which the function u tends to a ; the function u vanishes identically on \mathbb{B}^n . It is easy to observe that the function $\phi(\epsilon) = (\log(\frac{1}{\epsilon}))^{p-1}$ satisfies the two conditions in [M].

Last, Miklyukov and Vuorinen in [MV] showed that if we define

$$I(\epsilon) = \int_{\|u(x)-a\|<\epsilon} \|\nabla u(x)\|^p dx.$$

If the integral $I(\epsilon)$ satisfies one of the conditions

$$(3) \quad \int_0^1 \left(\frac{1}{I'(\epsilon)}\right)^{\frac{1}{p-1}} d\epsilon = \infty;$$

or

$$(4) \quad \int_0^1 \left(\frac{\epsilon}{I(\epsilon)}\right)^{\frac{1}{p-1}} d\epsilon = \infty;$$

or there exists a nonnegative function $f(\epsilon)$ satisfying conditions

$$(5) \quad I(\epsilon) \leq \epsilon^p (f(\epsilon))^{p-1},$$

for every $0 < \epsilon < \frac{1}{2}$; and

$$(6) \quad \sum_{k=0}^{\infty} \frac{1}{f(2^{-k})} = \infty;$$

or

$$(7) \quad \liminf_{\epsilon \rightarrow 0} \frac{I(\epsilon)}{\epsilon^p} < \infty;$$

then again the function u is identically equal to 0.

Again, it is not difficult to show that this result generalizes the one in [M].

In this paper we are going to give a more general condition on the integral $\int_{\|u(x)-a\|<\epsilon} \|\nabla u(x)\|^p dx$ under which we will still be able to deduce that the function u vanishes identically and this condition is more general than the ones appearing in [M] and [MV]. Let us remark here that the results of Koskela, Mizuta and Miklyukov and Vuorinen are for real valued functions, while our result is going to be more in the spirit of the initial results of M. and F. Riesz, Beurling and Jenkins, where they considered functions from the complex plane into the complex plane.

In that spirit, our results will hold for functions defined in the unit ball of \mathbb{R}^n into \mathbb{R}^n . Later, we will see how this general result includes the results in [K], [M] and [MV]. In this section we will introduce several definitions that will be needed in the rest of the paper. Let us start by recalling the definition of monotone function (in this paper we consider only continuous monotone functions).

Definition 1.1. *Let $\Omega \subset \mathbb{R}^n$ be an open set. A continuous function $u: \Omega \rightarrow \mathbb{R}$ is monotone (in the sense of Lebesgue) if*

$$\max_D u(x) = \max_{\partial D} u(x)$$

and

$$\min_D u(x) = \min_{\partial D} u(x)$$

hold whenever D is a domain with compact closure $\bar{D} \subset \Omega$.

The Sobolev space $W^{1,p}(\mathbb{B}^n)$ is defined in [HKM, Chapter 1]. It consists of functions $u: \mathbb{B}^n \rightarrow \mathbb{R}^n$ that have first distributional derivatives ∇u such that

$$\int_{\mathbb{B}^n} (|u(x)|^p + |\nabla u(x)|^p) dx < \infty.$$

The p -capacity better suited to our problem is the relative first order variational p -capacity defined also in [HKM, Chapter 2]. We will occasionally need the Sobolev class

$$ACL^p(\mathbb{B}^n) = \left\{ u \in ACL(\mathbb{B}^n) \text{ such that } \int_{\mathbb{B}^n} |\nabla u(x)|^p dx < \infty \right\}.$$

Here $ACL(\mathbb{B}^n)$ is the class of functions absolutely continuous on almost every line. These functions are continuous and their gradients are Borel functions. See for example [Vä, §26]. Smooth functions are dense in $W^{1,p}(\mathbb{B}^n)$, see [K]. In particular $ACL^p(\mathbb{B}^n)$ is dense in $W^{1,p}(\mathbb{B}^n)$.

It was proved in [MV1] that:

Theorem 1.3. *Let u be a continuous monotone function in $W^{1,p}(\mathbb{B}^n)$. Suppose that $n - 1 < p \leq n$. Let E be the set on the boundary of the unit ball where the non-tangential limit of u does not exist, then E has p -capacity zero.*

The proof is based on the modulus method after we obtained the following extension of Lindelöf's theorem.

Theorem 1.4. *Let u be a continuous monotone function in $W^{1,p}(\mathbb{B}^n)$. Suppose that $n-1 < p \leq n$. Then, for any $\epsilon > 0$, there exists an open set U in \mathbb{R}^n satisfying $\text{cap}_p(U) < \epsilon$ such that for any $x_0 \in \partial\mathbb{B}^n \setminus U$ and γ any curve ending at x_0 in \mathbb{B}^n with*

$$\lim_{x \rightarrow x_0, x \in \gamma} u(x) = \alpha,$$

then $u(x)$ has non-tangential limit α at x_0 .

The limitation $p > n - 1$ appears in a module estimate on $(n - 1)$ -dimensional spheres.

§2. Preliminaries and Oscillation Estimate.

Let us continue with some standard notation that will be used throughout the paper. The open ball centered at x_0 with radius r is denoted by $B^n(x_0, r)$. By $c(\alpha, \beta, \dots)$ we denote a constant that depends only on the parameters α, β, \dots and that may change value from line to line.

Let Γ be a family of curves in \mathbb{R}^n . Denote by $\mathcal{F}(\Gamma)$ the collection of admissible metrics for Γ . These are nonnegative Borel measurable functions $\rho: \mathbb{R}^n \rightarrow \mathbb{R} \cup \{\infty\}$ such that

$$\int_{\gamma} \rho \, ds \geq 1$$

for each locally rectifiable curve $\gamma \in \Gamma$. For $p \geq 1$ the weighted p -module of Γ is defined by

$$M_p(\Gamma) = \inf_{\rho \in \mathcal{F}(\Gamma)} \int_{\mathbb{R}^n} \rho^p \, dx.$$

If $\mathcal{F}(\Gamma) = \emptyset$, we set $M_p(\Gamma) = \infty$.

Upper bounds for moduli are obtained by testing with a particular admissible metric.

Before we state our main result in this paper, let us recall some definitions.

Let $\Omega \subset \mathbb{R}^n$, $n \geq 2$, be a domain and $F: \Omega \rightarrow \mathbb{R}^n$ be a mapping in the Sobolev space $W_{\text{loc}}^{1,n}(\Omega; \mathbb{R}^n)$ of functions in $L_{\text{loc}}^n(\Omega; \mathbb{R}^n)$ whose distributional derivatives belong to $L_{\text{loc}}^n(\Omega; \mathbb{R}^n)$. We can think of F as a deformation of some material whose initial configuration is Ω , and we seek some functional $I(F)$ representing the (non-linear) elastic energy whose minimum is attained at F , see [B1,2,3] and [S]. The differential of F at a point x is denoted by $DF(x)$, its norm is

$$|DF(x)| = \sup\{|DF(x)h|: h \in \mathbb{R}^n, |h| = 1\}$$

and its Jacobian determinant is $J_F(x) = \det DF(x)$. We assume that F is orientation preserving, meaning that $J_F(x) \geq 0$ for a. e. $x \in \Omega$. The *dilatation* of F at the point x is defined by the ratio

$$K(x) = \frac{|DF(x)|^n}{J_F(x)}.$$

If $K(x) \in L^\infty(\Omega; \mathbb{R}^n)$, then F is said to be a quasiregular mapping.

We will say that F is a mapping of *finite dilatation* if

$$1 \leq K(x) < \infty \text{ for a. e. } x \in \Omega;$$

that is, except for a set of measure zero in Ω , if $J_F(x) = 0$ then $DF(x) = 0$.

From now on we will assume that $K \in L^{n-1}$ and our domain Ω will be the unit ball \mathbb{B}^n of \mathbb{R}^n .

Definition 1.5. Let $F: \mathbb{B}^n \rightarrow \mathbb{R}^n$ be a mapping. We define the multiplicity function of F at some point $y \in \mathbb{R}^n$ with respect to some domain $D \subset \mathbb{B}^n$ as

$$N(F, D, y) = \#\{x \in D: F(x) = y\}.$$

Let us state our main result.

Theorem 1.6. Let F be a continuous mapping in the Sobolev space $W_{\text{loc}}^{1,n}(\mathbb{B}^n; \mathbb{R}^n)$, of finite dilatation in $W^{1,p}(\mathbb{B}^n)$ for $n-1 < p$. Let us assume also that the dilatation function $K(x) \in L^p$ for some $p > n-1$. Let E the set on the boundary of \mathbb{B}^n where the radial limit exist and are equal to a . Let $\mathbb{B}_\epsilon = \{y: \|y\| < \epsilon\}$, and $h(r)$ be a real function, a constant c independent of ϵ and an $\epsilon_0 > 0$ such that

$$N(F, \mathbb{B}^n, y) \leq ch(\|y\|)$$

for any $0 < \|y\| \leq \epsilon_0$. then if either

$$\lim_{m \rightarrow \infty} \frac{1}{m^p} \left(\int_{e^{-(m+1)}}^{e^{-m}} \frac{h(r)}{r} dr \right)^{\frac{p}{n}} = 0,$$

or

$$\sup_{\mathbb{B}_\epsilon} \left(\int_{\mathbb{B}_\epsilon} N(y, \mathbb{B}^n, F) dy \right) \left(\int_{\mathbb{B}_\epsilon} N(y, \mathbb{B}^n, F)^{\frac{1}{1-n}} dy \right)^{n-1} < C,$$

for any $0 < \epsilon < \epsilon_0$. Then if E has positive variational p -capacity then the mapping F is identically equal to a .

Observe that by [VG] mappings in the Sobolev space $W_{\text{loc}}^{1,n}(\mathbb{B}^n; \mathbb{R}^n)$ of finite dilatation their component functions are monotone functions.

Before we pass to the proof of our result, let us examine how it is related to the results of Koskela, Mizuta and Miklyukov and Vuorinen. Let us observe, first, that their covering condition on \mathbb{B}_ϵ is an integral condition involving the gradient of the real function u to some power p . For example, Koskela's condition requires that

$$\int_{\mathbb{B}_\epsilon} \|\nabla u\|^p dm \leq C \epsilon^p \left(\log \frac{1}{\epsilon} \right)^{p-1},$$

for $1 < p \leq n$. Since the function u is a real function, this covering condition is for an interval $(-\epsilon, \epsilon)$ about zero, therefore this condition should be replaced in the case of mappings from \mathbb{B}^n into \mathbb{R}^n by an integral condition on some power of the norm of the differential matrix $\|DF\|$ or the Jacobian J_F .

As we will see later, the two extra conditions we impose on the mapping F come naturally, first the integrability (in L^{n-1}) of the dilatation function $K(x)$ and second the monotonicity of the components of the mapping F . Yet, the second condition can be removed if we consider that the limits at the set $E \subset \partial\mathbb{B}^n$ are fine boundary limits. Loosely speaking, the norm of the gradient of the functions u in the work of Koskela, Mizuta and Miklyukov and Vuorinen should be replaced by $J_F^{\frac{1}{n}}$ in the case of a mapping F and thus, Koskela condition on \mathbb{B}_ϵ translates in this case to show that (we will assume from now on that the mapping F is sense preserving, that is $J_F \geq 0$ a.e. in \mathbb{B}^n)

$$\int_{\mathbb{B}_\epsilon} J_F(x)^{\frac{p}{n}} dm(x) \leq C \epsilon^p \left(\log \frac{1}{\epsilon} \right)^{p-1}.$$

Let us quickly show here that this is the case under the hypothesis of our theorem on the multiplicity function $N(y, \mathbb{B}^n, F)$. We are trying to find an upper bound of the integral

$$\int_{\mathbb{B}_\epsilon} J_F(x)^{\frac{p}{n}} dm(x) \leq C \left(\int_{\mathbb{B}_\epsilon} J_F(x) dm(x) \right)^{\frac{p}{n}},$$

where we have applied Holder's inequality. By the properties of the mapping F , we have that the following change of variable formula holds,

$$\int_{\mathbb{B}_\epsilon} J_F(x) dm(x) = \int_{y: \|y\| < \epsilon} N(y, \mathbb{B}^n, F) dm(y).$$

Therefore, by the theorem's condition on $N(y, \mathbb{B}^n, F)$ we have that

$$\int_{\mathbb{B}_\epsilon} J_F(x) dm(x) \leq \int_{y: \|y\| < \epsilon} h(r = \|y\|) dm(y),$$

and by choosing our function conveniently i.e. $h(r) = \left(\log \frac{1}{r} \right)^\delta$ for some positive δ conveniently chosen, we obtain Koskela's condition. In a similar way we can obtain Mizuta's and Miklyukov and Vuorinen's.

Mizuta's theorem in [M], states that if we have a function ϕ positive and nonincreasing on the interval $(0, \infty)$ with the properties that;

$$(1) \quad A^{-1} \phi(r) \leq \phi(r^2) \leq A \phi(r)$$

for every $r > 0$ and A a positive constant and

$$(2) \quad \int_0^1 [\phi(r)]^{\frac{-1}{(p-1)}} r^{-1} dr = \infty,$$

for some $1 < p < \infty$. Then, the uniqueness result follows if we have a Koskela's type condition

$$(3) \quad \int_{\mathbb{B}_\epsilon} \|\nabla u(x)\|^p dm(x) \leq C \epsilon^p \phi(\epsilon)$$

for any positive ϵ . To show that Mizuta's result follows from ours we'll need to show that if we take his function ϕ to be our function h and we impose his two conditions on h then the above inequality follows from ours. Mizuta's condition (2) for ϕ is equivalent by Ohtsuka [O] to condition (2) in our theorem if we take $N(\|y\|, \mathbb{B}^n, F) = h(\|y\|)$, that is, Mizuta's condition (2) follows from the multiplicity function being an A_n weight.

Let us pass to show that if h satisfies Mizuta's conditions (1) and (2) then inequality (3) follows with $\|\nabla u\|$ replaced by J_F when we go from real functions u to mappings F . Thus we need to show that

$$\begin{aligned} \int_{\mathbb{B}_\epsilon} J_F(x)^{\frac{p}{n}} dm(x) &\leq C \left(\int_{\mathbb{B}_\epsilon} J_F(x) dm(x) \right)^{\frac{p}{n}} \\ &= \left(\int_{y: \|y\| < \epsilon} N(y, \mathbb{B}^n, F) dm(y) \right)^{\frac{p}{n}}. \end{aligned}$$

Using the fact that $N(\|y\|, \mathbb{B}^n, F) = h(\|y\|)$, we need to show that

$$\left(\int_{y: \|y\| < \epsilon} h(\|y\|) dm(y) \right)^{\frac{p}{n}} \leq C \epsilon^p h(\epsilon),$$

and we are done. For this, we write the above integral as

$$\begin{aligned} \int_{y: \|y\| < \epsilon} h(\|y\|) dm(y) &= \int_0^\epsilon h(r) r^{n-1} dr \\ &= \sum_{j=0}^{\infty} \int_{\epsilon^{2^{j+1}}}^{\epsilon^{2^j}} h(r) r^{n-1} dr. \end{aligned}$$

Now, since the function h is nonincreasing and satisfies condition (1) we have that

$$\leq \sum_{j=0}^{\infty} h(\epsilon^{2^j}) \frac{(\epsilon^{2^j})^n - (\epsilon^{2^{j+1}})^n}{n} \leq \sum_{j=0}^{\infty} h(\epsilon) A^j \frac{(\epsilon^{2^j})^n - (\epsilon^{2^{j+1}})^n}{n}$$

$$= h(\epsilon) \left[\epsilon^n + (A-1) (\epsilon^n)^2 \sum_{k=0}^{\infty} (A \epsilon^{2n})^k \right].$$

The last series in the above equality is a geometric series whose ratio $A \epsilon^{2n}$ is less than one, since we have freedom in our choice of ϵ , thus we have that

$$\begin{aligned} \int_{\mathbb{B}_\epsilon} J_F(x)^{\frac{p}{n}} dm(x) &\leq C h(\epsilon) \left(\epsilon^n + (A-1) (\epsilon^n)^2 C \right)^{\frac{p}{n}} \\ &\leq C h(\epsilon) \left(\epsilon^n \right)^{\frac{p}{n}} = C h(\epsilon) \epsilon^p, \end{aligned}$$

as we wanted to show.

As for Miklyukov and Vuorinen's result, it is not difficult to show that condition (4) (for $p=n$) in their theorem is the same as condition (2) in our result (the A_n weight condition for the multiplicity function of $F N(y, \mathbb{B}^n, F)$, this follows from the Corollary in page 197 of Ohtsuka's paper [Oh].

Also, with a condition on J_F as in our theorem conditions (5) and (6) in [MV] follow. We showed above how condition (5) follows, we will show now how condition (6) follows, that is

$$\sum_{k=0}^{\infty} \frac{1}{h(2^{-k})} = \infty.$$

For that, let us start as in Koskela, we assume that $\phi(r) = h(r)^{p-1}$, thus by condition (2) in [K] we have that

$$\int_0^1 \frac{1}{h(r) r} dr = \infty$$

since the function $h(r)$ is nonincreasing we can rewrite the above integral as follows,

$$\begin{aligned} \infty &= \int_0^1 \frac{1}{h(r) r} dr = \infty = \sum_{k=0}^{\infty} \int_{2^{-(k+1)}}^{2^{-k}} \frac{1}{h(r) r} dr \\ &\leq \sum_{k=0}^{\infty} \frac{1}{h(2^{-k})} \int_{2^{-(k+1)}}^{2^{-k}} \frac{1}{r} dr = C \sum_{k=0}^{\infty} \frac{1}{h(2^{-k})}, \end{aligned}$$

and condition (6) in [MV] follows.

Observe also, that condition (2) in Mizuta's paper is equivalent to the necessary and sufficient condition in the Corollary in page 197 of Ohtsuka [Oh] once we make our choice of $\phi(r) = h(r) = N(r, \mathbb{B}^n, F)$.

After all of this, we have informally shown that our result Theorem 1.6 generalizes all the previous results related to the M. and F. Riesz's uniqueness theorem in two directions, namely:

1) Our results is for mappings.

2) When restricted to functions, i.e. $|\nabla u|$ replaced by $J_F^{\frac{1}{p}}$ we recuperate all the known results by Koskela, Mizuta and Miklyukov and Vuorinen.

Remark 1. If we are given a function u as in Koskela [K], the question will be how to extend it to a mapping F by finding its n component functions u_i , $i = 1, \dots, n$ in such a way that the conditions on the component functions of F given by Koskela will guarantee the condition in our theorem for the mapping F . Obviously the choice of the component functions of the mapping F has to be related to the function u . The question is, in which way?.

Let us examine now the monotonicity condition on our result. In the theorem we state that our function “approaches” a fixed point a on a set E on the boundary $\partial\mathbb{B}^n$. We require then monotonicity, to be able to say that for any rectifiable curve γ in \mathbb{B}^n ending at a point in E : either the limit of the mapping F along this curve does not exist or else, the limit exists and is equal to the same fixed point a . The reason we are able to conclude that is because as we mentioned before, for the class of mappings we are considering a Lindelöf theorem holds, as can be seen in [MV1] module sets of p -capacity 0 on $\partial\mathbb{B}^n$.

If now, we define the term “approaching” as in [Z], that is, fine boundary limit exist and is equal to the fixed point a in \mathbb{R}^n , then we will show (it will take some work in terms of delicate estimates, that the same result holds without the assumption that the components of the mapping F have to be monotone. We will show this in section 4.

§3. Point-wise behavior of weighted Sobolev functions

For the proof of Theorem 1.7 we are going to need the following lemma,

Lemma 2.1. *Under the hypotheses of Theorem 1.7 if we consider the weight $w_1(x) = (\ln \ln \frac{1}{\|F(x)\|})^{n-1}$, then there exist a positive constant C independent of p , the point x_0 , w_1 and r , such that*

$$\frac{1}{r^p} \int_{\mathbb{B}^n(x_0, r)} w_1(x) dx \leq (p, w_1) - \text{cap}(\mathbb{B}^n(x_0, r); \mathbb{B}^n(x_0, 2r)).$$

Proof. By a result in [HKM] all we need to show is that the measure μ defined as $d\mu(x) = w_1(x) dx$ satisfies a Poincare-type inequality. Namely, for each $\eta(x) \in C_0^\infty(\mathbb{B}^n(x_0, 2r))$ we need to show that

$$\int_{\mathbb{B}^n(x_0, 2r)} \eta^{n-1}(x) d\mu(x) \leq C r^{n-1} \int_{\mathbb{B}^n(x_0, 2r)} \|\nabla \eta(x)\|^{n-1} d\mu(x),$$

where C is a constant as in the statement of the lemma.

In [MV1] it was proved that the function $\eta(x) (\ln \ln \frac{1}{\|F(x)\|})$ is in the class $W_0^{1, n-1}(\mathbb{B}^n(x_0, 2r))$. This implies that this function satisfies a Poincare type inequality with respect to the euclidean metric. Thus, we have that

$$\begin{aligned} & \int_{\mathbb{B}^n(x_0, 2r)} \eta^{n-1}(x) (\ln \ln \frac{1}{\|F(x)\|})^{n-1} dx \\ & \leq C r^{n-1} \int_{\mathbb{B}^n(x_0, 2r)} \|\nabla(\eta(x) (\ln \ln \frac{1}{\|F(x)\|}))\|^{n-1} dx. \end{aligned}$$

Applying the product rule to the right hand side of the above inequality we obtain two terms, namely

$$\begin{aligned} & \int_{\mathbb{B}^n(x_0, 2r)} \eta^{n-1}(x) (\ln \ln \frac{1}{\|F(x)\|})^{n-1} dx \\ & \leq C r^{n-1} \int_{\mathbb{B}^n(x_0, 2r)} \|\nabla \eta(x)\|^{n-1} (\ln \ln \frac{1}{\|F(x)\|})^{n-1} dx \\ & \quad + \int_{\mathbb{B}^n(x_0, 2r)} \eta(x)^{n-1} |\nabla (\ln \ln \frac{1}{\|F(x)\|})|^{n-1} dx. \end{aligned}$$

By a result in [MV1] we can bound the second term inside the braces above as

$$\int_{\mathbb{B}^n(x_0, 2r)} \eta(x)^{n-1} |\nabla (\ln \ln \frac{1}{\|F(x)\|})|^{n-1} dx$$

$$\leq C \left(\int_{\mathbb{B}^n(x_0, 2r)} \|\nabla \eta(x)\|^n K(x)^{n-1} dx \right)^{\frac{n-1}{n}} \\ \left(\int_{\mathbb{B}^n(x_0, 2r)} K(x)^{n-1} dx \right)^{\frac{1}{n}},$$

since $K(x) \in L^{n-1}$ and $\eta(x) \in C_0^\infty(\mathbb{B}^n(x_0, 2r))$, it easily follows that among the two terms on the right hand side of the above inequality, the significant one is the first, thus we have that

$$\int_{\mathbb{B}^n(x_0, 2r)} \eta^{n-1}(x) \left(\ln \ln \frac{1}{\|F(x)\|} \right)^{n-1} dx \\ \leq C r^{n-1} \int_{\mathbb{B}^n(x_0, 2r)} \|\nabla \eta(x)\|^{n-1} \left(\ln \ln \frac{1}{\|F(x)\|} \right)^{n-1} dx$$

which constitutes the desired Poincare inequality.

Theorem 1.7. *Let F be a continuous mapping in the Sobolev space $W_{\text{loc}}^{1,n}(\mathbb{B}^n; \mathbb{R}^n)$, of finite dilatation in $W^{1,p}(\mathbb{B}^n)$. Let $\mathbb{B}_\epsilon = \{y: \|y\| < \epsilon\}$, and $h(r)$ be a real function and an $\epsilon_0 > 0$ such that*

$$N(F, \mathbb{B}^n, |y|) = h(\|y\|)$$

for any $0 < \|y\| \leq \epsilon_0$. Then if the dilatation function $K(x) \in L^{n-1}$, F is discrete and open if and only if

$$\int_0^{\epsilon_0} \frac{1}{r h(r)^{\frac{1}{n-1}} \left(\ln \ln \frac{1}{\|y\|} \right)^n} dr = \infty,$$

provided that

$$\sup_{\mathbb{B}_\epsilon} \left(\int_{\mathbb{B}_\epsilon} N(|y|, \mathbb{B}^n, F) \left(\ln \ln \frac{1}{\|y\|} \right)^n dy \right) \\ \left(\int_{\mathbb{B}_\epsilon} N(|y|, \mathbb{B}^n, F)^{\frac{1}{1-n}} \left(\ln \ln \frac{1}{\|y\|} \right)^{\frac{n}{1-n}} dy \right)^{n-1} < C,$$

for any $0 < \epsilon < \epsilon_0$.

Proof. One of the implications follows from The corollary in page 197 of [Mi] and the equivalence between the weighted p - modulus and the corresponding (p, w_1) -variational capacity where $w_1(y) = \left(\ln \ln \frac{1}{\|F(x)\|} \right)^{n-1}$.

The other implication is more complicated to proof. Without loss of generality we can assume that $a = 0$, thus we need to show that under our hypotheses $F^{-1}\{0\} = E$ is a discrete set, and this will follow immediately if we could show

that $M_{n-1}^{w_1}(\Lambda(E)) = 0$, since then this will imply that the 1-Hausdorff measure of E is equal to 0 and thus the set E is discrete and by a standard result of Titus and Young continuity of F will follow. Here $\Delta = \Lambda(E)$ consists of the family of rectifiable curves in \mathbb{R}^n ending at a point in E .

Let us pass to show that $M_{n-1}^{w_1}(\Lambda(E)) = 0$. It follows from a result in [VG] that if the mapping F is of finite dilatation then its components are monotone and thus using a result in [MV] the class of mapping under consideration satisfy a Lindelöf type theorem, which allows to say that $F(E) = \{0\}$. Consider now $\Delta_1 = \Lambda\{0\}$ to be the set of all the rectifiable curves ending at 0. Let ρ be an admissible metric for the family Δ_1 , then the metric $C (\rho \circ F)(x) \|DF(x)\|$ is admissible for the family Δ for a suitable constant C depending only on the dimension n . It is also immediate by Lindelöf's theorem that if we have a rectifiable curve $\gamma \in \Delta$ then $F \circ \gamma$ belongs to the family Δ_1 . Let us see this with some detail.

Let $\gamma \in \Delta$ such that $\gamma = \{(x_1(t), \dots, x_n(t)) : t \in [a, b]\}$ then $F \circ \gamma$ is given in parametric equations by $\{(y_1(t), \dots, y_n(t)) : y_i(t) = F_i(x_1(t), \dots, x_n(t)) ; i = 1, \dots, n\}$. If we denote by s the arclength parameter for γ and by \bar{s} the arclength parameter for the curve $F \circ \gamma$, a simple exercise on the chain rule in several variables gives us that they are related by the following formula

$$d\bar{s}(y) \leq ds(x) \sqrt{\left[\sum_{i=1}^n \left(\frac{\partial F_1}{\partial x_i} \right)^2 + \dots + \sum_{i=1}^n \left(\frac{\partial F_n}{\partial x_i} \right)^2 \right]},$$

and since $\sum_{i=1}^n \left(\frac{\partial F_j}{\partial x_i} \right)^2 \leq \|DF(x)\|^2$ for each $j = 1, \dots, n$ we obtain that

$$d\bar{s}(y) \leq \sqrt{n} ds(x) \|DF(x)\|.$$

Thus our constant $C = \sqrt{n}$. By the definition of the modulus of order p we have that

$$M_p^{w_1}(\Delta) \leq \int C^p (\rho \circ F)^p(x) \left(\ln \ln \frac{1}{\|F(x)\|} \right)^{n-1} \|DF(x)\|^p dm(x),$$

multiplying and dividing by $K(x)^{\frac{p}{n}}$ we have that

$$M_p^{w_1}(\Delta) \leq \int C^p (\rho \circ F)^p(x) \left(\ln \ln \frac{1}{\|F(x)\|} \right)^{n-1} \|DF(x)\|^p K(x)^{\frac{p}{n}} \frac{1}{K(x)^{\frac{p}{n}}} dm(x),$$

applying Hölder's inequality and using the fact that $J_F(x) = \frac{\|DF(x)\|^n}{K(x)}$ we obtain

$$M_p^{w_1}(\Delta) \leq \left(\int C (\rho \circ F)^n(x) \left(\ln \ln \frac{1}{\|F(x)\|} \right)^{\frac{(n-1)n}{p}} J_F(x) dm(x) \right)^{\frac{p}{n}}$$

$$\left(\int K(x)^{\frac{p}{n-p}} dm(x) \right)^{\frac{n-p}{n}}.$$

Since $J_F(x) > 0$ a.e. by assumption, we can use the formula for the change of variable to obtain that

$$M_p^{w_1}(\Delta) \leq \left(\int C \rho^n(y) N(|y|, \mathbb{B}^n, F) \left(\ln \ln \frac{1}{\|y\|} \right)^{\frac{(n-1)n}{p}} dm(x) \right)^{\frac{p}{n}}$$

$$\left(\int K(x)^{\frac{p}{n-p}} dm(x) \right)^{\frac{n-p}{n}}.$$

We now take $p = n - 1$, thus

$$M_{n-1}^{w_1}(\Delta) \leq \left(\int C (\rho \circ F)^n(x) N(|y|, \mathbb{B}^n, F) \left(\ln \ln \frac{1}{\|F(x)\|} \right)^n dm(x) \right)^{\frac{n-1}{n}} \left(\int K(x)^{n-1} dm(x) \right)^{\frac{1}{n}},$$

taking the infimum over all the admissible metrics *rho* for Δ_1 we obtain that

$$M_p^{w_1}(\Delta) \leq C (M_n^w(\Delta_1))^{\frac{n-1}{n}} \left(\int K(x)^{n-1} dm(x) \right)^{\frac{1}{n}},$$

where $w(y) = N(|y|, \mathbb{B}^n, F) \left(\ln \ln \frac{1}{\|y\|} \right)^n$. The hypotheses of our theorem guaranty by the corollary in page 197 of [Mi] that $M_n^w(\Delta_1) = 0$ and since we have also assumed that $K(x) \in L^{n-1}$ this implies that $M_{n-1}^{w_1}(\Delta) = 0$ which automatically implies that the $(n-1, w_1)$ -variational capacity of E is equal to 0.

Our aim next, will be to show that this implies that the linear measure of E is equal to 0 from which the discreteness of E will follow.

Let us denote by $H_1(E)$ the 1-Hausdorff measure of the set E . We want to show that $H_1(E) = 0$. By definition,

$$H_1(E) = \lim_{\delta \rightarrow 0} \left[\inf \left\{ \sum r_i : E \subset \bigcup \mathbb{B}^n(x_i, r_i), 0 < r_i < \delta \right\} \right].$$

Thus, applying Lemma 2.1 for $p = n - 1$ and $w_1(x) = \left(\ln \ln \frac{1}{\|F(x)\|} \right)^{n-1}$ we obtain that

$$\frac{1}{r^{n-1}} \int_{\mathbb{B}^n(x_0, r)} \left(\ln \ln \frac{1}{\|F(x)\|} \right)^{n-1} dx \leq (n-1, w_1) - \text{cap}(\mathbb{B}^n(x_0, r); \mathbb{B}^n(x_0, 2r)).$$

Let $\mathbb{B}^n(0, \delta)$ be defined as the ball centered at 0 and radius δ , and let $\Omega_\delta = F^{-1}(\mathbb{B}^n(0, \delta))$. Let us take a ring completely contained in Ω_δ centered at $x_0 \in E$ and

defined by the concentric balls $\mathbb{B}^n(x_0, r)$ and $\mathbb{B}^n(x_0, 2r)$ as shown in the diagram below

Observe that $|F(x)| \leq \delta$ for any $x \in \Omega_\delta$. Therefore, we have the following inequality

$$\frac{1}{r^{n-1}} \int_{\mathbb{B}^n(x_0, r)} \left(\ln \ln \frac{1}{\delta}\right)^{n-1} dx \leq \frac{1}{r^{n-1}} \int_{\mathbb{B}^n(x_0, r)} \left(\ln \ln \frac{1}{\|F(x)\|}\right)^{n-1} dx.$$

Computing the term in the left hand side of the above inequality we obtain

$$C r \left(\ln \ln \frac{1}{\delta}\right)^{n-1} \leq \frac{1}{r^{n-1}} \int_{\mathbb{B}^n(x_0, r)} \left(\ln \ln \frac{1}{\|F(x)\|}\right)^{n-1} dx,$$

where C is a universal constant. Let us define the function $h(x_0, r)$ as follows

$$h(x_0, r) = \frac{1}{r^{n-1}} \int_{\mathbb{B}^n(x_0, r)} \left(\ln \ln \frac{1}{\|F(x)\|}\right)^{n-1} dx.$$

With this new notation, we have that

$$C r \left(\ln \ln \frac{1}{\delta}\right)^{n-1} \leq \frac{h(x_0, r)}{r}.$$

Observe that when $\delta \rightarrow 0$ we have that $x_0 \rightarrow E$ and $r \rightarrow 0$. So we have that

$$\lim_{\delta \rightarrow 0} \frac{h(x_0, r)}{r} = \infty.$$

That is, for any positive ϵ we can find a positive β such that $r < \epsilon h(x_0, r)$ whenever $0 < r < \beta$ and x_0 is close enough to the set E . Let $\{\mathbb{B}^n(x_i, r_i): x_i \in E \text{ and } 0 < r_i < \beta\}_i$ be a covering of the set E . If we define $H_1^\delta = \inf\{\sum r_i: E \subset \bigcup \mathbb{B}^n(x_i, r_i), 0 < r_i < \delta\}$, where without loss of generality we can assume that all the x_i 's are in E , we have that

$$H_1^\delta \leq \sum_i r_i \leq \sum_i \epsilon h(x_i, r_i) \leq \epsilon \sum_i (n-1, w_1) - \text{cap}(\mathbb{B}^n(x_i, r_i); \mathbb{B}^n(x_i, 2r_i)).$$

We know that $(n-1, w_1) - \text{cap}(E) = 0$. Hence by the definition of variational capacity and using rings to cover the set E rather than balls (observe that we can always assume that both rings and balls are centered at points in E), then for any positive $\tilde{\epsilon}$ we can find a covering of E by rings $E \subset \bigcup_i (\mathbb{B}^n(x_i, r_i); \mathbb{B}^n(x_i, 2r_i))$ such that

$$\sum_i (n-1, w_1) - \text{cap}(\mathbb{B}^n(x_i, r_i); \mathbb{B}^n(x_i, 2r_i)) \leq (n-1, w_1) - \text{cap}(E) + \tilde{\epsilon}.$$

Choosing that covering in the previous inequality we have that

$$H_1^\delta \leq \epsilon [(n-1, w_1) - \text{cap}(E) + \tilde{\epsilon}] = \epsilon \tilde{\epsilon}$$

and since both ϵ and $\tilde{\epsilon}$ are arbitrary, letting $\delta \rightarrow 0$ we obtain that $H_1(E) = 0$ as we wanted to show.

In particular, $E = F^{-1}\{0\}$ can not contain any segment, and therefore it is a totally disconnected set. Replacing $F(x)$ by $F(x) - b$ it follows that $F^{-1}\{b\}$ is totally disconnected for any $b \in \mathbb{R}^n$. The mapping F is thus an orientation preserving light mapping and it follows from a theorem of Titus and Young [TY] that F is open and discrete and theorem 1.7 is proved.

□

§4. Proof of Theorem 1.6

We now pass to prove Theorem 1.6.

Proof of Theorem 1.6.

Let ρ be an admissible metric for $\Delta_1 = \Lambda(\{0\})$, then the metric $\tilde{\rho}(x) = C \rho(F(x)) |DF(x)|$ is admissible for the family $\Delta = \Lambda(E)$ for a suitable constant C depending only on the dimension n , this follows as in Theorem 1.7.

By the definition of the modulus of order p we have that

$$M_p(\Delta) \leq \int C^p (\rho \circ F)^p(x) \|DF(x)\|^p dm(x),$$

multiplying and dividing by $K(x)^{\frac{p}{n}}$ we have that

$$M_p(\Delta) \leq \int C^p (\rho \circ F)^p(x) \|DF(x)\|^p K(x)^{\frac{p}{n}} \frac{1}{K(x)^{\frac{p}{n}}} dm(x),$$

applying Hölder's inequality and using the fact that $J_F(x) = \frac{\|DF(x)\|^n}{K(x)}$ we obtain

$$M_p(\Delta) \leq \left(\int C (\rho \circ F)^n(x) J_F(x) dm(x) \right)^{\frac{p}{n}} \left(\int K(x)^{\frac{p}{n-p}} dm(x) \right)^{\frac{n-p}{n}}.$$

Since $J_F(x) > 0$ a.e. by assumption, we can use the formula for the change of variable to obtain that

$$M_p(\Delta) \leq \left(\int C \rho^n(y) N(|y|, \mathbb{B}^n, F) dm(x) \right)^{\frac{p}{n}} \left(\int K(x)^{\frac{p}{n-p}} dm(x) \right)^{\frac{n-p}{n}}.$$

Choose now for large positive m the admissible metrics $\rho(y) = \rho_m(y)$ as follows; $\rho_m(y) = \frac{1}{m |y|}$ whenever $e^{-(m+1)} < |y| \leq e^{-m}$ and 0 otherwise. Thus we have

$$M_p(\Delta) \leq \left(\int_{e^{-(m+1)} < |y| \leq e^{-m}} C \frac{1}{m^n |y|^n} N(|y|, \mathbb{B}^n, F) dm(x) \right)^{\frac{p}{n}} \left(\int K(x)^{\frac{p}{n-p}} dm(x) \right)^{\frac{n-p}{n}},$$

using the hypothesis of the theorem $N(|y|, \mathbb{B}^n, F) \leq C h(|y|)$ we have that

$$M_p(\Delta) \leq \frac{1}{m^p} \left(\int_{e^{-(m+1)}}^{e^{-m}} C \frac{h(|y|)}{|y|} dm(x) \right)^{\frac{p}{n}}$$

$$\left(\int K(x)^{\frac{p}{n-p}} dm(x) \right)^{\frac{n-p}{n}}.$$

It is clear that the exponent of the dilatation function $K(x)$ in the formula above $\frac{p}{n-p}$ is some $\tilde{p} > n - 1$, so we choose our p such that $\frac{p}{n-p} = \tilde{p}$, hence the second factor on the right hand side of the above inequality is finite. Now, letting $m \rightarrow \infty$ we obtain by one of the hypothesis of our theorem that $M_p(\Delta) = 0$. It is well known that this implies that the variational p -capacity of E is equal to 0 and the theorem is proved.

□

§5. Concluding remarks

Finally in this section, we are going to explore further the results of this paper when applied to some known classes of mappings. We will start with the quasiregular mappings.

It is well known that a mapping F is quasiregular if $K(x)$ is an L^∞ function, thus $1 \leq K(x) \leq K < \infty$ a.e. x , and in this case $\frac{1}{C} J_F(x) \leq |DF(x)| \leq C J_F(x)$ a.e. x for some constant independent of x and we don't have to make any further assumption on the dilatation function. Our theorems 1.6 and 1.7 will then read as follows;

Theorem 5.1. *Let F be a continuous mapping in the Sobolev space $W_{\text{loc}}^{1,n}(\mathbb{B}^n; \mathbb{R}^n)$, of bounded dilatation. Let $\mathbb{B}_\epsilon = \{y: \|y\| < \epsilon\}$, and $h(r)$ be a real function and an $\epsilon_0 > 0$ such that*

$$N(F, \mathbb{B}^n, |y|) = h(\|y\|)$$

for any $0 < \|y\| \leq \epsilon_0$. F is discrete and open if and only if

$$\int_0^\epsilon \frac{1}{r h(r)^{\frac{1}{n-1}} (\ln \ln \frac{1}{\|y\|})^n} dr = \infty,$$

provided that

$$\sup_{\mathbb{B}_\epsilon} \left(\int_{\mathbb{B}_\epsilon} N(|y|, \mathbb{B}^n, F) (\ln \ln \frac{1}{\|y\|})^n dy \right)$$

$$\left(\int_{\mathbb{B}_\epsilon} N(|y|, \mathbb{B}^n, F)^{\frac{1}{1-n}} (\ln \ln \frac{1}{\|y\|})^{\frac{n}{1-n}} dy \right)^{n-1} < C,$$

for any $0 < \epsilon < \epsilon_0$.

and

Theorem 5.2. *Let F be a continuous mapping in the Sobolev space $W_{\text{loc}}^{1,n}(\mathbb{B}^n; \mathbb{R}^n)$, of bounded dilatation. Let E the set on the boundary of \mathbb{B}^n where the radial limit exist and are equal to a . Let $\mathbb{B}_\epsilon = \{y: \|y\| < \epsilon\}$, and $h(r)$ be a real function, a constant c independent of ϵ and an $\epsilon_0 > 0$ such that*

$$N(F, \mathbb{B}^n, y) \leq ch(\|y\|)$$

for any $0 < \|y\| \leq \epsilon_0$. then if either

$$\lim_{m \rightarrow \infty} \frac{1}{m^p} \left(\int_{e^{-(m+1)}}^{e^{-m}} \frac{h(r)}{r} dr \right)^{\frac{p}{n}} = 0,$$

or

$$\sup_{\mathbb{B}_\epsilon} \left(\int_{\mathbb{B}_\epsilon} N(y, \mathbb{B}^n, F) dy \right) \left(\int_{\mathbb{B}_\epsilon} N(y, \mathbb{B}^n, F)^{\frac{1}{1-n}} dy \right)^{n-1} < C,$$

for any $0 < \epsilon < \epsilon_0$. Then if E has positive variational p -capacity then the mapping F is identically equal to a .

let us now talk about different choices of the function $h(|y|)$ which will satisfy the hypotheses of our theorems,

which will clearly validate them. As we mentioned in previous sections, by choosing

$$h(|y|) = h(r) = \left(\ln \frac{1}{|y|} \right)^\delta$$

for small values of r and suitable choices of δ , we recover all the results in [K], [Mi1] and [MV]. For

instance choosing δ such that $\frac{\delta p}{n} = p - 1$ we recover the results in [K] and thus in

[Mi1] and a straightforward computation will show that the function

$$h(|y|) = \left(\ln \frac{1}{|y|} \right)^{\frac{n(p-1)}{p}}$$

satisfies the conditions in Theorem 1.6.

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