Test 2:

Show all work for credit.

1. (10 pts.) Prove that \( \frac{1}{2} + \frac{2}{4} + \frac{3}{8} + \ldots + \frac{n}{2^n} = \frac{2^{n-1} - 2 - n}{2^n} \) for all \( n \geq 1 \).

\[ P(1): \frac{1}{2} = \frac{2^2 - 2 - 1}{2^1} \checkmark \]

\[ P(k) \rightarrow P(k+1) \]

\[ \frac{1}{2} + \frac{2}{4} + \frac{3}{8} + \ldots + \frac{k}{2^k} + \frac{k+1}{2^{k+1}} = \frac{2^{k+1} - 2 - k}{2^{k+1}} \]

\[ = \frac{2^{k+2} - 2 - (k+1)}{2^{k+1}} \checkmark \]

2. (10 pts.) Prove that \((n^3 + 3n^2 + 2n)\) is divisible by 3 for all \( n \geq 1 \).

\[ P(1): 1^3 + 3(1)^2 + 2(1) = 6 \text{ is divisible by 3.} \]

\[ P(k) \rightarrow P(k+1) \]

\[ (k+1)^3 + 3(k+1)^2 + 2(k+1) = k^3 + 3k^2 + 3k + 1 + 3k^2 + 6k + 3 + 3k + 3 \]

\[ \text{Div by 3} \]

\[ \frac{3k^3 + 9k + 9}{3} \]

3. (10 pts.) Let \( a_1 = 2 \), \( a_2 = 9 \), and \( a_n = 2a_{n-1} + 3a_{n-2} \) for \( n \geq 3 \)

a) Calculate \( a_3, a_4, a_5 \)

\[ a_3 = 2 \cdot 9 + 3 \cdot 2 = 24 \]

\[ a_4 = 2 \cdot 24 + 8 = 48 + 27 = 75 \]

\[ a_5 = 2 \cdot 75 + 3 \cdot 24 = 150 + 72 = 222 \]

b) Show that \( a_n \leq 3^n \) for all positive integers \( n \).

\[ P(1): 2 \leq 3 \checkmark \text{ True} \]

\[ P(n): 3^n \leq 3^n \]

\[ P(k) \rightarrow P(k+1) \]

\[ a_{k+1} \leq 3^k, 1 \leq k < n \]

\[ a_n = 2a_{n-1} + 3a_{n-2} \leq 2 \cdot 3^{n-1} + 3 \cdot 3^{n-2} = 2 \cdot 3^{n-1} + 3 \cdot 3^{n-2} = 3 \cdot 3^{n-1} = 3^n \]

\[ a_{n+1} = 2a_n + 3a_{n-2} \leq 2 \cdot 3 + 3 \cdot 3^{n-2} = 2 \cdot 3 + 3 \cdot 3^{n-2} = 3 \cdot 3^{n-2} = 3^n \]

\[ a_{n+1} = 2a_n + 3a_{n-2} \leq 2 \cdot 3 + 3 \cdot 3^{n-2} = 2 \cdot 3 + 3 \cdot 3^{n-2} = 3 \cdot 3^{n-2} = 3^n \]
4. (6 pts.) Use a factorial argument to prove that \( C(n+4, n-1) = C(n+3, n-1) + C(n+3, n-2) \).

\[
C(n+4, n-1) = \frac{(n+4)!}{(n-1)! \cdot (n+4-n+1)!} = \frac{(n+4)!}{(n-1)! \cdot 5!} = \frac{(n+4)(n+3)(n+2)(n+1)n(n-1)!}{(n-1)! \cdot 5!}.
\]

\[
C(n+3, n-1) = \frac{(n+3)!}{(n-1)! \cdot 4!} = \frac{(n+3)(n+2)(n+1)n\cdot 5}{5!} = \sum_n = \frac{4! \cdot 5}{5!}.
\]

\[
C(n+3, n-2) = \frac{(n+3)!}{(n-2)! \cdot 5!} = \frac{(n+3)(n+2)(n+1)n(n-1)!}{5!} = \frac{(n+4)(n+3)(n+2)(n+1)n \cdot 5}{5!}.
\]

5. (9 pts.) In the questions below suppose that a “word” is any string of seven letters of the alphabet, with repeated letters allowed.

(a) How many words begin with A or end with B (not both)?

\[
\begin{align*}
A & \Rightarrow 1 \times 26^3 \times 25^2 \times 7 \sum \{50 \times 26^5 = 594,068,800 \}
\end{align*}
\]

(b) How many words begin with A or B or end with A or B (not both)?

\[
2 \left[ 2 \times 26^5 \times 24 \right] = 96 \times 26^5 = 1,140,612,096
\]

(c) How many words have exactly one vowel?

\[
2 \times 7 \times 21^6 = 3,001,814,235
\]

6. (4 pts.) In the questions below consider all bit strings of length 12.

(a) How many begin with 110?

\[
2^9 = 512
\]

(b) How many have exactly four 1s?

\[
\binom{12}{4} = 495
\]
7. In the questions below nine people (Ann, Ben, Cal, Dot, Ed, Fran, Gail, Hal, and Ida) are in a room. Five of them stand in a row for a picture.

a) (2 pts.) In how many ways can this be done if both Ed and Gail are in the picture?

\[ 5! \cdot P(7, 3) = 4500 \]

\[ a) \quad 4500 \]

b) (2 pts.) In how many ways can this be done if Ed and Gail are in the picture, standing next to each other?

\[ 2! \cdot 4! \cdot P(7, 3) = 1680 \]

\[ b) \quad 1680 \]

c) (2 pts.) In how many ways can this be done if Ann and Ben are in the picture, but not standing next to each other?

\[ 5! \cdot P(7, 3) - 2 \cdot 4 \cdot P(7, 3) \]

\[ = 4500 - 1680 = 2820 \]

\[ c) \quad 2820 \]

d) (2 pts.) In how many ways can all nine be seated in a row, if four of them do not want to sit next to one another?

\[ 5! \cdot P(6, 4) = 43200 \]

\[ d) \quad 43200 \]

e) (2 pts.) In how many ways can all nine be seated in a circle, if three of them do not want to sit next to one another?

\[ 5! \cdot P(6, 3) = 14400 \]

\[ e) \quad 14400 \]
8. In the questions below a club with 20 women and 17 men needs to form a committee of size six.

a) (3 pts.) How many committees are possible if the committee must have at least two women?

\[
\frac{\binom{20}{2} \binom{17}{4} + \binom{20}{3} \binom{17}{3} + \binom{20}{4} \binom{17}{2} + \binom{20}{5} \binom{17}{1} + \binom{20}{6}}{\binom{37}{6}} - \frac{\binom{37}{6} - \binom{20}{5}}{\binom{6}{5}} = \frac{2,182,648}{\binom{37}{6}} - \frac{2}{\binom{6}{5}} = 2,182,648 \quad \text{a)}
\]

b) (3 pts.) How many committees are possible if the committee must consist of at most four men?

\[
\binom{20}{0} \binom{17}{6} = 2,182,648 \quad \text{b)}
\]

c) (3 pts.) How many committees are possible if three of the club members refuse to work with one another?

\[
\frac{\binom{24}{3} + \binom{24}{5}}{\binom{30}{6}} = \frac{1,344,904 + 834,768}{2,179,672} = \frac{2,182,672}{2,179,672} = 1,134 \quad \text{c)}
\]

9 (3 pts.) A computer randomly prints three-digit codes, with no repeated digits in any code (for example, 387, 072, 760). What is the minimum number of codes that must be printed in order to guarantee that at least six of the codes are identical?

\[
N = 5(720) + 1 = 3,601
\]
10. (3 pts.) State and use the binomial theorem to prove the following:

\[ 3^{100} = \binom{100}{0} + \binom{100}{1} \cdot 2^1 + \binom{100}{2} \cdot 2^2 + \binom{100}{3} \cdot 2^3 + \ldots + \binom{100}{99} \cdot 2^{99} + \binom{100}{100} \cdot 2^{100}. \]

\[ (x+y)^n = \sum_{j=0}^{\infty} \binom{n}{j} x^{n-j} y^j \]

Let \( x=1, y=2 \), \( n=100 \)

\[ \binom{100}{0} \cdot 1^{100-0} \cdot 2^0 = \sum_{j=0}^{100} \binom{100}{j} \cdot 1^{n-j} \cdot 2^j \]

11. (3 pts.) Find the coefficient of \( x^7y^6 \) in the expansion of \((2x-y)^{11}\).

\[ \binom{11}{6} \cdot (2)^5 \cdot (-1)^6 = 14,784 \]

12. (6 pts.) Complete the truth table for the proposition \( \neg (r \rightarrow \neg q) \lor (p \land \neg r) \).

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In the questions below write the negation of the statement in good English. Don’t write “It is not true that...”

13. (3 pts.) Some skiers do not speak Swedish and go to Sweden.

\[ \text{All skiers speak Swedish or do not go to Sweden.} \]

14. (3 pts.) Write the negation of the statement.

\[ \exists x \forall y (\neg A(x) \land T(x,y)) \]

\[ \forall x \exists y (A(x) \lor \neg T(x,y)) \]

15. (4 pts.) Consider the following theorem: If \( n \) is an even integer, then \( n + 1 \) is odd. Give a proof by contraposition of this theorem.

Suppose \( n + 1 \) is even. Therefore, \( n + 1 = 2k \Rightarrow \]
\[ n = 2k - 1 = 2(k-1) + 1, \text{ which is odd} \]
\[ \text{ (contradictory with assumption) } \]
16. (7 pts.) Determine whether the following argument is valid:

She is a Math Major or a Computer Science Major.

If she does not know discrete math, she is not a Math Major.

If she knows discrete math, she is smart.

She is not a Computer Science Major.

Therefore, she is smart.

\[
\begin{align*}
(p \lor q) \land \sim q & \rightarrow p \\
\sim r \rightarrow \sim p & \equiv p \rightarrow r \\
\text{but } r \rightarrow s
\end{align*}
\]

Also, can show:

\[
(p \lor q) \land (\sim r \rightarrow \sim p) \land (r \rightarrow s) \land \sim q \rightarrow s
\]

is a tautology.